Cosmological nonlinear hydrodynamics with post-Newtonian corrections

Jai-chan Hwang[(a)], Hyerim Noh[(b)], and Dirk Puetzfeld[(c)]
[(a)] *Department of Astronomy and Atmospheric Sciences, Kyungpook National University, Taegu, Korea*
[(b)] *Korea Astronomy and Space Science Institute, Daejon, Korea*
[(c)] *Department of Physics and Astronomy, Iowa State University, Ames, IA, 50011, USA*

arXiv:astro-ph/0507085 v1 5 Jul 2005

The purpose of this paper is to present general relativistic cosmological hydrodynamic equations in Newtonian-like forms using the post-Newtonian (PN) method. The PN approximation, based on the assumptions of weak gravitational fields and slow motions, provides a way to estimate general relativistic effects in the fully nonlinear evolution stage of the large-scale cosmic structures. We extend Chandrasekhar's first order PN (1PN) hydrodynamics based on the Minkowski background to the Robertson-Walker background. We *assume* the presence of Friedmann's cosmological spacetime as a background. In the background we include the three-space curvature, the cosmological constant and general pressure; we show that our 1PN approach is successful only for spatially flat cosmological background. In the Newtonian order and 1PN order we include general pressure, stress, and flux. The Newtonian hydrodynamic equations appear naturally in the 0PN order. The spatial gauge degree of freedom is fixed in a unique manner and the basic equations are arranged without taking the temporal gauge condition. In this way we can conveniently try alternative temporal gauge conditions. We investigate a number of temporal gauge conditions under which all the remaining variables are equivalently gauge-invariant. Our aim is to present the fully nonlinear 1PN equations in a form suitable for implementation in conventional Newtonian hydrodynamic simulations with minimal extensions. The 1PN terms can be considered as relativistic corrections added to the well known Newtonian equations. The proper arrangement of the variables and equations in combination with suitable gauge conditions would allow the possible future 1PN cosmological simulations to become more tractable. Our equations and gauges are arranged for that purpose. We suggest ways of controlling the numerical accuracy. The typical 1PN order terms are about $10^{-6} \sim 10^{-4}$ times smaller than the Newtonian terms. However, we cannot rule out possible presence of cumulative effects due to the time-delayed propagation of the relativistic gravitational field with finite speed, in contrast to the Newtonian case where changes in the gravitational field are felt instantaneously. The quantitative estimation of such effects is left for future numerical simulations.

Contents

I. INTRODUCTION

The nonlinear evolution of large-scale cosmic structures is usually investigated within the framework of Newtonian theory either in analytic studies or numerical simulations. Without investigating general relativistic effects, however, it is not clear whether the Newtonian theory is sufficient to handle such large-scale cosmological structures. If we regard Einstein's gravity theory to be the correct framework on such cosmic scales, the relativistic effects should exist always. The point is to which level we can practically ignore relativistic correction terms considering currently available levels of observations and experiments. In case relativistic or nonlinear effects are not large, we have two widely known ways to estimate relativistic effects: one is the perturbation approach [1–3], and the other one is the post-Newtonian (PN) approximation [4–9].

In the perturbation study, we recently have investigated the weakly nonlinear regimes based on Einstein's gravity. We showed that, except for the gravitational wave coupling, the relativistic scalar-type perturbations coincide exactly with the Newtonian ones up to the second order [10]. The pure relativistic correction terms appear in the third order. These terms turn out to be independent of the horizon scale and small ($\sim 10^{-5}$ order) compared with the second-order Newtoninan/relativistic terms [11]. Our studies have shown, from the view point of Einstein's gravity, that Newtonian gravity is practically reliable near the horizon scale where structures are supposed to be weakly nonlinear. Therefore, justifying its use in current large-scale numerical simulations which cover the Hubble volume. Considering the action-at-a-distance nature of Newton's gravity our result is a rather surprising one because the relativistic perturbation theory is applicable on *all scales* including the super-horizon scale. Thus, our perturbation study ensures that to the weakly nonlinear order Newtonian theory is reliable even near and beyond the horizon scale.

Now, how about situation in the nonlinear regime on scales smaller than the current horizon? On such scales the structures could be in a fully nonlinear stage. However, if the relativistic gravity effect is small we can apply another approximation scheme which is well developed to handle isolated bodies in Einstein's theory: the PN approximation. The PN approximation has been important to test Einstein's gravity. It presents the relativistic equations in a form similar to the well known Newtonian equations. In the PN method, by assuming that the relativistic effects are small, we expand relativistic corrections in powers of v/c. In nearly virialized systems we have $GM/(Rc^2) \sim (v/c)^2$. Thus, the PN approximation is applicable in the slow motion and weak gravitational field regime. The first order PN (1PN) approximation corresponds to adding relativistic correction terms of the order $(v/c)^2$ to the Newtonian order terms. In this approach we recover the well known Newtonian hydrodynamics equations in the 0PN order [5]. The PN approximation is suitable to study systems in which Newtonian gravity has a dominant role, while the relativistic effects are small but non-negligible. Well known applications include the precession of Mercury's perihelion, and various solar system tests of Einstein's gravity theory like the light deflection [12]. Recent successful applications include the generation of gravitational waves from compact binary objects, and the weakly relativistic evolution stages of isolated systems of celestial bodies [13]. Another important application is relativistic celestial mechanics which is required by recent technology driven precise measurements of the solar system bodies [14,15]. Notice that all these applications are based on the PN approximation of isolated systems assuming the Minkowski background spacetime.

In this work on the cosmological PN approximation we assume the presence of a Robertson-Walker background. Thus, we have the Friedmann world model built in as the background spacetime. We will show that the Newtonian hydrodynamic equations come out naturally in the 0PN order which is the Newtonian limit. In the context of large-scale cosmic structures PN effects could be essentially important in handling gravitational waves and gravitational lensing effects. It is well known that Newtonian order treatment of the gravitational lensing shows only half of the result from Einstein's theory which is due to ignoring the 1PN correction in the metric. We also anticipate that PN correction terms could affect the dynamic evolution of large-scale structures, especially considering the action-at-a-distance nature of the Newtonian theory. This can be compared with the relativistic situation in which the gravitational field propagates with finite speed. In this work we present the complete set of equations to study 1PN effects in a cosmological context. The PN approach can be compared with our previous studies of the weakly nonlinear regime based on the perturbation approach which is applicable in the fully relativistic regime and on all scales. In comparison, although the PN equations are applicable in weak gravity regions inside the horizon, these are applicable in fully nonlinear situation. Thus, the two approaches are complementary in enhancing our understanding of the relativistic evolutionary aspects of the large-scale structure in the universe.

In the PN approximation we attempt to study the equations of motion and the field equations in Einstein's gravity in a Newtonian way as closely as possible. The Newtonian and post-Newtonian equations of motion follow from the energy-momentum conservation. The Newtonian order potential and 1PN order metric variables are determined by Einstein's equations in terms of the Newtonian matter and potential variables. Thus, the metric (or relativistic) contributions are reinterpreted as small correction terms to the well known Newtonian hydrodynamic equations. The Newtonian equations naturally follow in the 0PN order. The form of 1PN equations is affected by our choice of the gauge conditions. In this work we take unique spatial gauge conditions which fix the spatial gauge modes completely; under these spatial gauge conditions the remaining variables are all equivalently spatially gauge-invariant, see Sec. V. The temporal gauge condition (slicing condition) will be deployed to handle the mathematical treatment of the equations conveniently. We show how to choose the temporal gauge condition which also removes the temporal gauge mode completely. Under each of such temporal gauge conditions the remaining variables are equivalently (spatially and temporally) gauge-invariant. We arrange the equations and gauges so that the final form of equations is suitable for numerical implementation in conventional cosmological hydrodynamic simulations.

We present basic quantities and the derivation of our 1PN equations in Sec. II and III. If the reader is more interested in the application of the 1PN equations, the basic set of cosmological 1PN equations is summarized in Sec. IV. The gauge issue is expounded in Sec. V. We discuss our results and several unresolved issues in Sec. VI.

II. BASIC QUANTITIES

A. Curvature

As the metric we take

$$\tilde{g}_{00} \equiv -\left[1 - \frac{1}{c^2}2U + \frac{1}{c^4}\left(2U^2 - 4\Phi\right)\right] + \mathcal{O}^{-6},$$
$$\tilde{g}_{0i} \equiv -\frac{1}{c^3}aP_i + \mathcal{O}^{-5},$$
$$\tilde{g}_{ij} \equiv a^2\left(1 + \frac{1}{c^2}2V\right)\gamma_{ij} + \mathcal{O}^{-4}, \tag{1}$$

where $x^0 \equiv ct$, and $a(t)$ is the cosmic scale factor of the background Friedmann world model. Indices $a, b, c, \dots$ indicate spacetime, and $i, j, k, \dots$ indicate space. Tildes indicate spacetime covariant quantities, i.e., spacetime indices of quantities with tilde are raised and lowered with the spacetime metric $\tilde{g}_{ab}$. The spatial index of P_i is based on γ_{ij} in raising and lowering indices with γ^{ij}, an inverse of γ_{ij}. γ_{ij} is the comoving (time-independent) spatial part of the Robertson-Walker metric, see Eq. (2) in [16]. In a flat Robertson-Walker background γ_{ij} becomes δ_{ij} if we take Cartesian coordinates. Compared with our previous notations used in perturbation studies in [17], we set the comoving part of the three-space background metric $g^{(3)}_{ij} \equiv \gamma_{ij}$ and use part of the Latin indices to indicate the space. We are following Chandrasekhar and Nutku's notation [5,7] extended to the cosmological situation, see also Fock [4]. The $2U/c^2$ term in $\tilde{g}_{00}$ gives the Newtonian limit, and if we ignore all the Newtonian and post-Newtonian correction terms we have the Robertson-Walker spacetime. Thus, our PN formulation is built on the cosmological background spacetime. $\mathcal{O}^{-n}$ indicates $(v/c)^{-n}$ and higher order terms that we ignore. The expansion in Eq. (1) is valid to 1PN order [5]. Dimensions are as follows

$$[\tilde{g}_{ab}] = [\tilde{g}^{ab}] = 1, \quad [\gamma_{ij}] = [\gamma^{ij}] = 1, \quad [a] = 1, \quad [c] = L\mathcal{T}^{-1}, \quad [U] = [V] = [c^2], \quad [P_i] = [P^i] = [c^3], \quad [\Phi] = [c^4], \tag{2}$$

where L and $\mathcal{T}$ indicate the length and the time dimensions, respectively.

In our metric convention in Eq. (1) we have ignored the possible presence of $\frac{1}{c^2}\left(2C_{,i|j} + C_{i|j} + C_{j|i}\right)$ like terms in $\tilde{g}_{ij}$ with $C^i_{\ |i} \equiv 0$ by choosing the spatial C-gauge conditions ($C \equiv 0 \equiv C_i$) which remove the spatial gauge mode completely; this will be explained in Sec. V; a vertical bar indicates the covariant derivative based on γ_{ij}. Under such spatial gauge conditions, we can regard all our remaining PN variables as equivalently spatially gauge-invariant ones, see Sec. V. We still have a freedom to take the temporal gauge condition which will be chosen later depending on the mathematical simplification or feasibility of physical interpretation of the problem under consideration, see Sec. V: in the perturbation approach we termed this a gauge-ready approach [18]. We also have ignored the spatially tracefree-transverse part of the metric (C_{ij} with $C^i_i \equiv 0 \equiv C^j_{i|j}$) because gravitational waves are known to show up only from the 2.5PN order [8].

The inverse metric becomes

$$\tilde{g}^{00} = -\left[1 + \frac{1}{c^2}2U + \frac{1}{c^4}\left(2U^2 + 4\Phi\right)\right] + \mathcal{O}^{-6},$$
$$\tilde{g}^{0i} = -\frac{1}{c^3}\frac{1}{a}P^i + \mathcal{O}^{-5},$$
$$\tilde{g}^{ij} = \frac{1}{a^2}\left(1 - \frac{1}{c^2}2V\right)\gamma^{ij} + \mathcal{O}^{-4}. \tag{3}$$

The determinant of the metric tensor $\tilde{g}$ is

$$\sqrt{-\tilde{g}} = a^3\sqrt{\gamma}\left[1 + \frac{1}{c^2}\left(3V - U\right) + \mathcal{O}^{-4}\right], \tag{4}$$

where γ is the determinant of γ_{ij}.

The connection is

$$\tilde{\Gamma}^0_{00} = -\frac{1}{c^3}\dot{U} + \frac{1}{c^5}\left(-2\dot{\Phi} + \frac{1}{a}P^iU_{,i}\right) + L^{-1}\mathcal{O}^{-7},$$
$$\tilde{\Gamma}^0_{0i} = -\frac{1}{c^2}U_{,i} - \frac{1}{c^4}\left(2\Phi_{,i} + \dot{a}P_i\right) + L^{-1}\mathcal{O}^{-6},$$
$$\tilde{\Gamma}^0_{ij} = a^2\left\{\frac{1}{c}\frac{\dot{a}}{a}\gamma_{ij} + \frac{1}{c^3}\left[\left(\dot{V} + 2\frac{\dot{a}}{a}\left(U+V\right)\right)\gamma_{ij} + \frac{1}{a}P_{(i|j)}\right]\right\} + L^{-1}\mathcal{O}^{-5},$$
$$\tilde{\Gamma}^i_{00} = \frac{1}{a^2}\left\{-\frac{1}{c^2}U^{,i} + \frac{1}{c^4}\left[2\left(U+V\right)U^{,i} - 2\Phi^{,i} - \left(aP^i\right)^{\cdot}\right]\right\} + L^{-1}\mathcal{O}^{-6},$$
$$\tilde{\Gamma}^i_{0j} = \frac{1}{c}\frac{\dot{a}}{a}\delta^i_j + \frac{1}{c^3}\left[\dot{V}\delta^i_j - \frac{1}{2a}\left(P^i{}_{|j} - P_j{}^{|i}\right)\right] + L^{-1}\mathcal{O}^{-5},$$
$$\tilde{\Gamma}^i_{jk} = \Gamma^{(\gamma)i}{}_{jk} + \frac{1}{c^2}\left(V_{,k}\delta^i_j + V_{,j}\delta^i_k - V^{,i}\gamma_{jk}\right) + L^{-1}\mathcal{O}^{-4}, \tag{5}$$

where $\Gamma^{(\gamma)i}{}_{jk}$ is the connection based on γ_{ij}; we introduce $P_{(i|j)} \equiv \frac{1}{2}\left(P_{i|j} + P_{j|i}\right)$ and $P_{[i|j]} \equiv \frac{1}{2}\left(P_{i|j} - P_{j|i}\right)$. We have $U_{,0} = \frac{1}{c}\frac{\partial U}{\partial t} \equiv \frac{1}{c}\dot{U}$.

The Riemann curvature is

$$\tilde{R}^0{}_{00i} = -\frac{1}{c^5}\left(\ddot{a}P_i + \frac{1}{a}U_{,i|j}P^j\right) + L^{-2}\mathcal{O}^{-7},$$
$$\tilde{R}^0{}_{0ij} = L^{-2}\mathcal{O}^{-6},$$
$$\tilde{R}^0{}_{i0j} = \frac{1}{c^2}\left(a\ddot{a}\gamma_{ij} + U_{,i|j}\right) + \frac{1}{c^4}\Bigg\{a^2\left[\dot{V} + 2\frac{\dot{a}}{a}\left(U+V\right)\right]^{\cdot}\gamma_{ij} + \left[-a\dot{a}\dot{U} + 2\dot{a}^2\left(U+V\right)\right]\gamma_{ij}$$
$$-2U_{,(i}V_{,j)} - U_{,i}U_{,j} + U^{,k}V_{,k}\gamma_{ij} + 2\Phi_{,i|j} + \left(aP_{(i|j)}\right)^{\cdot}\Bigg\} + L^{-2}\mathcal{O}^{-6},$$
$$\tilde{R}^0{}_{ijk} = \frac{1}{c^3}a^2\left[2\left(\dot{V} + \frac{\dot{a}}{a}U\right)_{,[j}\gamma_{k]i} - \frac{1}{a}P_{[j|k]i} + \frac{2K}{a}\gamma_{i[j}P_{k]}\right] + L^{-2}\mathcal{O}^{-5},$$
$$\tilde{R}^i{}_{00j} = \frac{1}{c^2}\left(\frac{\ddot{a}}{a}\delta^i_j + \frac{1}{a^2}U^{,i}{}_{|j}\right) + \frac{1}{c^4}\Bigg\{\left[\ddot{V} + \frac{\dot{a}}{a}\left(\dot{U} + 2\dot{V}\right)\right]\delta^i_j - \frac{1}{a^2}\Big[2\left(U+V\right)U^{,i}{}_{|j}$$
$$+U^{,i}U_{,j} + U^{,i}V_{,j} + U_{,j}V^{,i} - U^{,k}V_{,k}\delta^i_j - 2\Phi^{,i}{}_{|j}\Big] + \frac{1}{2a^2}\left[a\left(P^i{}_{|j} + P_j{}^{|i}\right)\right]^{\cdot}\Bigg\} + L^{-2}\mathcal{O}^{-6},$$
$$\tilde{R}^i{}_{0jk} = \frac{1}{c^3}\left[2\left(\dot{V} + \frac{\dot{a}}{a}U\right)_{,[j}\delta^i_{k]} - \frac{1}{a}P_{[j|k]}{}^i\right] + L^{-2}\mathcal{O}^{-5},$$
$$\tilde{R}^i{}_{j0k} = \frac{1}{c^3}\left[\dot{V}_{,j}\delta^i_k - \dot{V}^{,i}\gamma_{jk} + \frac{\dot{a}}{a}\left(U_{,j}\delta^i_k - U^{,i}\gamma_{jk}\right) + \frac{1}{2a}\left(P^i{}_{|j} - P_j{}^{|i}\right)_{|k}\right] + L^{-2}\mathcal{O}^{-5},$$
$$\tilde{R}^i{}_{jkl} = -2K\gamma_{j[k}\delta^i_{l]} + \frac{1}{c^2}2\left(-\dot{a}^2\gamma_{j[k}\delta^i_{l]} + V_{,j|[k}\delta^i_{l]} - V^{,i}{}_{|[k}\gamma_{l]j}\right) + L^{-2}\mathcal{O}^{-4}. \tag{6}$$

It is convenient to have

$$P^i{}_{|jk} \equiv P^i{}_{|kj} - R^{(\gamma)i}{}_{ljk}P^l, \quad P_{i|jk} = P_{i|kj} + R^{(\gamma)l}{}_{ijk}P_l,$$
$$R^{(\gamma)i}{}_{jkl} = K\left(\delta^i_k\gamma_{jl} - \delta^i_l\gamma_{jk}\right), \quad R^{(\gamma)}_{ij} = 2K\gamma_{ij}, \quad R^{(\gamma)} = 6K, \tag{7}$$

where K indicates the comoving (time-independent) part of background spatial curvature with dimension $[K] = L^{-2}$. The Ricci curvature and the scalar curvature become

$$\tilde{R}^0_0 = \frac{1}{c^2}\left(3\frac{\ddot{a}}{a} + \frac{\Delta}{a^2}U\right) + \frac{1}{c^4}\left\{3\ddot{V} + 3\frac{\dot{a}}{a}\left(\dot{U} + 2\dot{V}\right) + 6\frac{\ddot{a}}{a}U - \frac{1}{a^2}\left[U^{,i}\left(U-V\right)_{,i} + 2V\Delta U - 2\Delta\Phi - \left(aP^i{}_{|i}\right)^{\cdot}\right]\right\}$$
$$+L^{-2}\mathcal{O}^{-6},$$
$$\tilde{R}^0_i = \frac{1}{c^3}\left[2\left(\dot{V} + \frac{\dot{a}}{a}U\right)_{,i} + \frac{1}{2a}\left(P^j{}_{|ji} - \Delta P_i - 2KP_i\right)\right] + L^{-2}\mathcal{O}^{-5},$$
$$\tilde{R}^i_0 = -\frac{1}{c^3}\frac{1}{a^2}\left[2\left(\dot{V} + \frac{\dot{a}}{a}U\right)^{,i} + \frac{1}{2a}\left(P^j{}_{|j}{}^i - \Delta P^i + 2KP^i\right)\right] + L^{-2}\mathcal{O}^{-5},$$
$$\tilde{R}^i_j = \frac{2K}{a^2}\delta^i_j + \frac{1}{c^2}\left[\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - \frac{\Delta + 4K}{a^2}V\right)\delta^i_j + \frac{1}{a^2}\left(U-V\right)^{|i}{}_j\right] + L^{-2}\mathcal{O}^{-4},$$
$$\tilde{R} = \frac{6K}{a^2} + \frac{1}{c^2}\left[6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) + 2\frac{\Delta}{a^2}U - 4\frac{\Delta + 3K}{a^2}V\right] + L^{-2}\mathcal{O}^{-4}, \tag{8}$$

where Δ is a Laplacian operator based on γ_{ij}.

The Weyl curvature is introduced as

$$\tilde{C}^a{}_{bcd} \equiv \tilde{R}^a{}_{bcd} - \frac{1}{2}\left(\delta^a_c\tilde{R}_{bd} - \delta^a_d\tilde{R}_{bc} + \tilde{g}_{bd}\tilde{R}^a_c - \tilde{g}_{bc}\tilde{R}^a_d\right) + \frac{1}{6}\tilde{R}\left(\delta^a_c\tilde{g}_{bd} - \delta^a_d\tilde{g}_{bc}\right). \tag{9}$$

To 1PN order we have

$$\tilde{C}^0{}_{00i} = -\frac{1}{c^3}\frac{K}{a}P_i + L^{-2}\mathcal{O}^{-5},$$
$$\tilde{C}^0{}_{0ij} = L^{-2}\mathcal{O}^{-6},$$
$$\tilde{C}^0{}_{i0j} = \frac{1}{c^2}\left[\frac{1}{2}\left(U+V\right)_{,i|j} - \frac{1}{6}\Delta\left(U+V\right)\gamma_{ij}\right] + L^{-2}\mathcal{O}^{-4},$$
$$\tilde{C}^0{}_{ijk} = \frac{1}{c^3}\frac{a}{2}\left[\left(P^l{}_{|l[k} - \Delta P_{[k} - 2KP_{[k}\right)\gamma_{j]i} - 2P_{[j|k]i}\right] + L^{-2}\mathcal{O}^{-5},$$
$$\tilde{C}^i{}_{00j} = \frac{1}{c^2}\frac{1}{a^2}\left[\frac{1}{2}\left(U+V\right)^{|i}{}_j - \frac{1}{6}\Delta\left(U+V\right)\delta^i_j\right] + L^{-2}\mathcal{O}^{-4},$$
$$\tilde{C}^i{}_{0jk} = \frac{1}{c^3}\frac{1}{2a}\left[\left(P^l{}_{|l[k} - \Delta P_{[k} + 2KP_{[k}\right)\delta^i_{j]} - 2P_{[j|k]}{}^i\right] + L^{-2}\mathcal{O}^{-5},$$
$$\tilde{C}^i{}_{j0k} = \frac{1}{c^3}\frac{1}{4a}\left[\left(P^l{}_{|l}{}^i - \Delta P^i + 2KP^i\right)\gamma_{jk} - \left(P^l{}_{|lj} - \Delta P_j + 2KP_j\right)\delta^i_k + 2\left(P^i{}_{|j} - P_j{}^{|i}\right)_{|k}\right] + L^{-2}\mathcal{O}^{-5},$$
$$\tilde{C}^i{}_{jkl} = \frac{1}{c^2}\left[\frac{2}{3}\Delta\left(U+V\right)\delta^i_{[k}\gamma_{l]j} + \left(U+V\right)_{,j|[k}\delta^i_{l]} - \left(U+V\right)^{,i}{}_{|[k}\gamma_{l]j}\right] + L^{-2}\mathcal{O}^{-4}. \tag{10}$$

We can check the tracefree nature of the Weyl tensor: $\tilde{C}^b{}_{bcd} = 0 = \tilde{C}^c{}_{bcd}$. Here we encounter non-vanishing traces involving K terms, like

$$\tilde{C}^b{}_{b0i} = -\frac{1}{c^3}\frac{K}{a}P_i + L^{-2}\mathcal{O}^{-5}, \quad \tilde{C}^c{}_{0ci} = -\frac{1}{c^3}\frac{2K}{a}P_i + L^{-2}\mathcal{O}^{-5}, \tag{11}$$

whereas

$$\tilde{C}^c{}_{0c0} = L^{-2}\mathcal{O}^{-4}, \quad \tilde{C}^b{}_{bij} = L^{-2}\mathcal{O}^{-4}, \quad \tilde{C}^c{}_{ic0} = L^{-2}\mathcal{O}^{-5}, \quad \tilde{C}^c{}_{icj} = L^{-2}\mathcal{O}^{-4}. \tag{12}$$

This indicates that the presence of K terms in the 1PN order appears to be not reliable. Later this point will be resolved as we show that the K terms indeed can be related to the higher order terms in the PN expansion, see below Eq. (88). Until we reach such a conclusion about the effect of K term we will keep it in our equations.

B. Energy-momentum tensor

The normalized fluid four-vector $\tilde{u}^a$ with $\tilde{u}^a \tilde{u}_a \equiv -1$ gives, to 1PN order,

$$\begin{aligned}
\tilde{u}^0 &= 1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + U\right) + \frac{1}{c^4}\left[\frac{3}{8}v^4 + v^2\left(\frac{3}{2}U + V\right) + \frac{1}{2}U^2 + 2\Phi - v^i P_i\right] + \mathcal{O}^{-6}, \\
\tilde{u}^i &\equiv \frac{1}{c}\frac{1}{a}v^i \tilde{u}^0, \\
\tilde{u}_0 &= -\left\{1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 - U\right) + \frac{1}{c^4}\left[\frac{3}{8}v^4 + v^2\left(\frac{1}{2}U + V\right) + \frac{1}{2}U^2 - 2\Phi\right]\right\} + \mathcal{O}^{-6}, \\
\tilde{u}_i &= a\left\{\frac{1}{c}v_i + \frac{1}{c^3}\left[v_i\left(\frac{1}{2}v^2 + U + 2V\right) - P_i\right]\right\} + \mathcal{O}^{-5}, \qquad (13)
\end{aligned}$$

where the index of v^i is based on γ_{ij}; v^i corresponds to the peculiar velocity field. The energy-momentum tensor is decomposed into fluid quantities as follows

$$\tilde{T}_{ab} = \tilde{\varrho}c^2\left(1 + \frac{1}{c^2}\tilde{\Pi}\right)\tilde{u}_a\tilde{u}_b + \tilde{p}\left(\tilde{u}_a\tilde{u}_b + \tilde{g}_{ab}\right) + \tilde{q}_a\tilde{u}_b + \tilde{q}_b\tilde{u}_a + \tilde{\pi}_{ab}. \qquad (14)$$

The energy density $\tilde{\mu}(\equiv \tilde{\varrho}c^2 + \tilde{\varrho}\tilde{\Pi})$ is decomposed into the material energy density $\tilde{\varrho}c^2$ and the internal energy density $\tilde{\varrho}\tilde{\Pi}$; $\tilde{p}$, $\tilde{q}_a$, and $\tilde{\pi}_{ab}$ are the isotropic pressure, the flux, and the anisotropic stress, respectively. We have $\tilde{q}_a\tilde{u}^a \equiv 0$, $\tilde{\pi}_{ab}\tilde{u}^b \equiv 0$, $\tilde{\pi}^c_c \equiv 0$, and $\tilde{\pi}_{ab} \equiv \tilde{\pi}_{ba}$, thus

$$\tilde{q}_0 = -\frac{1}{c}\frac{1}{a}\tilde{q}_i v^i, \quad \tilde{\pi}_{0i} = -\frac{1}{c}\frac{1}{a}\tilde{\pi}_{ij}v^j, \quad \tilde{\pi}_{00} = \frac{1}{c^2}\frac{1}{a^2}\tilde{\pi}_{ij}v^i v^j. \qquad (15)$$

We introduce

$$\tilde{q}_i \equiv \frac{1}{c}aQ_i, \quad \tilde{\pi}_{ij} \equiv a^2\Pi_{ij}, \qquad (16)$$

where indices of Q_i and Π_{ij} are based on γ_{ij}. We take $\tilde{q}_a$ and $\tilde{\pi}_{ab}$ to have post-Newtonian orders as introduced in Eq. (16), see for example [19]. This will be justified by the energy conservation equation in the Newtonian context which will be derived later, see Eq. (102). In a strictly single component situation we can always remove Q_i by following the fluid element. However, there exist situations where we have the additional flux terms present even when we follow the fluid elements; the fundamental origin of such flux terms can be traced to the presence of additional components in the energy-momentum. Thus, in our case it is more general and convenient to keep the flux terms separately. Up to the 1PN order the condition $\tilde{\pi}^c_c \equiv 0$ gives

$$\Pi^i_i = \frac{1}{c^2}\Pi_{ij}v^i v^j. \qquad (17)$$

We also set

$$\tilde{\varrho} \equiv \varrho, \quad \tilde{\Pi} \equiv \Pi, \quad \tilde{p} \equiv p. \qquad (18)$$

Dimensions are as follows

$$\begin{aligned}
&[\tilde{u}_a] = [\tilde{u}^a] = 1, \quad [\tilde{T}_{ab}] = [\tilde{T}^a_b] = [\tilde{T}^{ab}] = [\tilde{p}] = [\tilde{q}_a] = [\tilde{q}^a] = [\tilde{\pi}_{ab}] = [\tilde{\varrho}c^2] = ML^{-1}\mathcal{T}^{-2}, \\
&[v_i] = [v^i] = [c], \quad [\Pi] = [c^2], \quad [Q_i] = [Q^i] = [c\tilde{q}_a] = [\varrho c^3], \quad [\Pi_{ij}] = [\Pi^i_j] = [\Pi^{ij}] = [\tilde{\pi}_{ab}] = [\varrho c^2]. \qquad (19)
\end{aligned}$$

We have

$$\begin{aligned}
\tilde{T}_{00} = \varrho c^2\Bigg\{&1 + \frac{1}{c^2}\left(v^2 - 2U + \Pi\right) + \frac{1}{c^4}\Bigg[v^2\left(v^2 + 2V + \Pi + \frac{p}{\varrho}\right) - 2U\Pi + 2U^2 - 4\Phi \\
&+ \frac{1}{\varrho}\left(2Q_i v^i + \Pi_{ij}v^i v^j\right)\Bigg] + \mathcal{O}^{-6}\Bigg\},
\end{aligned}$$

$$\tilde{T}_{0i} = -a\varrho c^2 \left\{ \frac{1}{c} v_i + \frac{1}{c^3} \left[v_i \left(v^2 + 2V + \Pi + \frac{p}{\varrho} \right) - P_i + \frac{1}{\varrho} \left(Q_i + \Pi_{ij} v^j \right) \right] + \mathcal{O}^{-5} \right\},$$

$$\tilde{T}_{ij} = a^2 \varrho c^2 \left\{ \frac{1}{c^2} \left(v_i v_j + \frac{p}{\varrho} \gamma_{ij} + \frac{1}{\varrho} \Pi_{ij} \right) + \frac{1}{c^4} \left[v_i v_j \left(v^2 + 2U + 4V + \Pi + \frac{p}{\varrho} \right) + 2V \frac{p}{\varrho} \gamma_{ij} - 2 v_{(i} P_{j)} + \frac{2}{\varrho} Q_{(i} v_{j)} \right] + \mathcal{O}^{-6} \right\},$$

$$\tilde{T} = -\varrho c^2 \left[1 + \frac{1}{c^2} \left(\Pi - 3 \frac{p}{\varrho} \right) + \mathcal{O}^{-6} \right]. \tag{20}$$

From Eq. (14) covariantly we have $\tilde{T} = -\tilde{\varrho} c^2 \left(1 + \frac{1}{c^2} \tilde{\Pi} \right) + 3\tilde{p}$.

C. Fluid-frame kinematic quantities

The projection tensor $\tilde{h}_{ab}$ based on the fluid-frame four-vector $\tilde{u}^a$ is defined as

$$\tilde{h}_{ab} \equiv \tilde{g}_{ab} + \tilde{u}_a \tilde{u}_b. \tag{21}$$

The kinematic quantities are defined as [20]

$$\tilde{\theta}_{ab} \equiv \tilde{h}_a^c \tilde{h}_b^d \tilde{u}_{c;d}, \quad \tilde{\theta} \equiv \tilde{u}^a{}_{;a}, \quad \tilde{\sigma}_{ab} \equiv \tilde{\theta}_{(ab)} - \frac{1}{3} \tilde{\theta} \tilde{h}_{ab}, \quad \tilde{\omega}_{ab} \equiv \tilde{\theta}_{[ab]}, \quad \tilde{a}_a \equiv \dot{\tilde{\tilde{u}}}_a \equiv \tilde{u}_{a;b} \tilde{u}^b, \tag{22}$$

where we have $\tilde{u}^a \tilde{\theta}_{ab} \equiv 0$, $\tilde{u}^a \tilde{\sigma}_{ab} \equiv 0$, $\tilde{u}^a \tilde{\omega}_{ab} \equiv 0$, $\tilde{u}^a \tilde{a}_a \equiv 0$, and $\tilde{\theta} \equiv \tilde{\theta}^a{}_a$. The kinematic quantities $\tilde{\theta}$, $\tilde{a}_a$, $\tilde{\sigma}_{ab}$, and $\tilde{\omega}_{ab}$ are the expansion scalar, the acceleration vector, the shear tensor, and the vorticity tensor, respectively.

To 1PN order, the projection tensor becomes

$$\tilde{h}_0^0 = -\frac{1}{c^2} v^2 - \frac{1}{c^4} \left[v^2 \left(v^2 + 2U + 2V \right) - v^i P_i \right] + \mathcal{O}^{-6},$$

$$\tilde{h}_i^0 = a \left\{ \frac{1}{c} v_i + \frac{1}{c^3} \left[v_i \left(v^2 + 2U + 2V \right) - P_i \right] \right\} + \mathcal{O}^{-5},$$

$$\tilde{h}_0^i = -\frac{1}{c} \frac{1}{a} v^i \left\{ 1 + \frac{1}{c^2} v^2 + \frac{1}{c^4} \left[v^2 \left(v^2 + 2U + 2V \right) - v^j P_j \right] \right\} + \mathcal{O}^{-7},$$

$$\tilde{h}_j^i = \delta_j^i + \frac{1}{c^2} v^i v_j + \frac{1}{c^4} \left[v^i v_j \left(v^2 + 2U + 2V \right) - v^i P_j \right] + \mathcal{O}^{-6}. \tag{23}$$

The kinematic quantities become

$$\tilde{\theta}_{ij} = \frac{1}{c} a \left(v_{i|j} + \dot{a} \gamma_{ij} \right) + \frac{1}{c^3} a \left\{ v_j \left(a v_i \right)^{\cdot} + v_{i|j} \left(\frac{1}{2} v^2 + U + 2V \right) + v^k \left(v_i v_{k|j} + v_j v_{i|k} \right) + a \left[\dot{V} + \frac{\dot{a}}{a} \left(U + 2V \right) + \frac{1}{a} v^k V_{,k} + \frac{1}{2} \frac{\dot{a}}{a} v^2 \right] \gamma_{ij} + 2 v_{[i} \left(U + V \right)_{,j]} - P_{[i|j]} \right\} + L^{-1} \mathcal{O}^{-5}, \tag{24}$$

$$\tilde{\theta} = \frac{1}{c} \left(3 \frac{\dot{a}}{a} + \frac{1}{a} v^i{}_{|i} \right) + \frac{1}{c^3} \left[3\dot{V} + 3 \frac{\dot{a}}{a} U + \frac{3}{a} V_{,i} v^i + \frac{1}{a} U v^i{}_{|i} + \frac{1}{2a^3} \left(a^3 v^2 \right)^{\cdot} + \frac{1}{2a} \left(v^2 v^i \right)_{|i} \right] + L^{-1} \mathcal{O}^{-5}, \tag{25}$$

$$\tilde{\sigma}_{ij} = \frac{1}{c} a \left(v_{(i|j)} - \frac{1}{3} v^k{}_{|k} \gamma_{ij} \right) + \frac{1}{c^3} a \left\{ v_{(i} \left(a v_{j)} \right)^{\cdot} + v_{(i|j)} \left(\frac{1}{2} v^2 + U + 2V \right) + v^k \left(v_{k|(i} v_{j)} + v_{(i} v_{j)|k} \right) - v_i v_j \left(\dot{a} + \frac{1}{3} v^k{}_{|k} \right) - \frac{1}{3} \gamma_{ij} \left[\frac{1}{2} a \left(v^2 \right)^{\cdot} + \left(U + 2V \right) v^k{}_{|k} + \frac{1}{2} \left(v^2 v^k \right)_{|k} \right] \right\} + L^{-1} \mathcal{O}^{-5}, \tag{26}$$

$$\tilde{\omega}_{ij} = \frac{1}{c} a v_{[i|j]} + \frac{1}{c^3} a \left\{ v_{[j} \left(a v_{i]} \right)^{\cdot} + v_{[i|j]} \left(\frac{1}{2} v^2 + U + 2V \right) + v^k \left(v_{k|[j} v_{i]} + v_{[j} v_{i]|k} \right) + 2 v_{[i} \left(U + V \right)_{,j]} - P_{[i|j]} \right\} + L^{-1} \mathcal{O}^{-5}, \tag{27}$$

$$\tilde{a}_i = -\frac{1}{c^2}U_{,i}\left(1+\frac{1}{c^2}v^2\right) - \frac{1}{c^4}2\Phi_{,i} + \left[1+\frac{1}{c^2}\left(\frac{1}{2}v^2+U\right)\right]\left(\frac{\partial}{\partial t}+\frac{1}{a}\mathbf{v}\cdot\nabla\right)\frac{1}{c}\tilde{u}_i + \frac{1}{c^4}\left(-v^2V_{,i}+v_jP^j{}_{|i}\right)$$
$$+L^{-1}\mathcal{O}^{-6}. \tag{28}$$

From $\tilde{u}^c\tilde{a}_c \equiv 0$ and $\tilde{u}^b\tilde{\theta}_{ab} \equiv 0$ we have

$$\tilde{a}_0 = -\frac{1}{c}\frac{1}{a}v^i\tilde{a}_i, \quad \tilde{\theta}_{0i} = -\frac{1}{c}\frac{1}{a}v^j\tilde{\theta}_{ij}, \quad \tilde{\theta}_{00} = \frac{1}{c^2}\frac{1}{a^2}v^iv^j\tilde{\theta}_{ij}, \tag{29}$$

and similarly for $\tilde{\sigma}_{0a}$ and $\tilde{\omega}_{0a}$.

The electric and the magnetic parts of the Weyl (conformal) tensor are introduced as

$$\tilde{E}_{ab} \equiv \tilde{C}_{acbd}\tilde{u}^c\tilde{u}^d, \quad \tilde{H}_{ab} \equiv \frac{1}{2}\tilde{\eta}_{ac}{}^{ef}\tilde{C}_{efbd}\tilde{u}^c\tilde{u}^d, \tag{30}$$

where $\tilde{\eta}^{abcd} \equiv \frac{1}{\sqrt{-\tilde{g}}}\epsilon^{abcd}$ is the totally antisymmetric tensor density with ϵ^{abcd}, the totally antisymmetric Levi-Civita symbol with $\epsilon^{0123} \equiv 1$. $\tilde{E}_{ab}$ and $\tilde{H}_{ab}$ are symmetric, tracefree and $\tilde{u}^a\tilde{E}_{ab} = 0 = \tilde{u}^a\tilde{H}_{ab}$. To 1PN order we have

$$\begin{aligned}\tilde{E}^i_j &= -\tilde{C}^i{}_{00j}\\ &= -\frac{1}{c^2}\frac{1}{a^2}\left[\frac{1}{2}(U+V)^{|i}{}_j - \frac{1}{6}\Delta(U+V)\delta^i_j\right] + L^{-2}\mathcal{O}^{-4}, \qquad (31)\\ \tilde{H}^i_j &= -\frac{1}{2}\tilde{u}^0\tilde{u}^0\left[\tilde{\eta}^i{}_{0l}{}^k\left(\tilde{C}^l{}_{k0j} - \frac{1}{c}\frac{1}{a}v^l\tilde{C}^0{}_{k0j}\right) + \tilde{\eta}^i{}_{lk}{}^0\frac{1}{c}\frac{1}{a}v^l\tilde{C}^k{}_{00j}\right]\\ &= \frac{1}{c^3}\frac{1}{2a^3}\eta^{ikl}\left\{\left[\frac{1}{2}\left(P^m{}_{|ml} - \Delta P_l + 2KP_l\right) + \frac{1}{3}v_l\Delta(U+V)\right]\gamma_{kj} + P_{l|kj} - v_l(U+V)_{|kj}\right\} + L^{-2}\mathcal{O}^{-5}, \qquad (32)\end{aligned}$$

where we introduced $\eta^{ijk} \equiv \frac{1}{\sqrt{\gamma}}\epsilon^{ijk}$ with ϵ^{ijk}, the totally antisymmetric Levi-Civita symbol with $\epsilon^{123} \equiv 1$. Thus $\tilde{\eta}^{0ijk} = \sqrt{\frac{\gamma}{-\tilde{g}}}\eta^{ijk}$. Indices of η^{ijk} are based on γ_{ij}. $\tilde{E}_{0a}$ follows from $\tilde{u}^b\tilde{E}_{ab} = 0$ which gives $\tilde{E}_{0a} = -(\tilde{u}^i/\tilde{u}^0)\tilde{E}_{ia} = -(v^i/ca)\tilde{E}_{ia}$, thus $\tilde{E}_{0i} \sim L^{-2}\mathcal{O}^{-3}$ and $\tilde{E}_{00} \sim L^{-2}\mathcal{O}^{-4}$. For nonvanishing K, $\tilde{H}_{ab}$ is not tracefree to $L^{-2}\mathcal{O}^{-3}$, but as we mentioned earlier the term involving K is already $\mathcal{O}^{-2}$ order higher, thus consistent.

D. Normal-frame kinematic quantities

The normalized normal-frame four-vector $\tilde{n}^a$, which is normal to the space-like hypersurfaces, is defined as $\tilde{n}_i \equiv 0$ with $\tilde{n}^a\tilde{n}_a \equiv -1$. To 1PN order we have

$$\begin{aligned}&\tilde{n}^0 = 1 + \frac{1}{c^2}U + \frac{1}{c^4}\left(\frac{1}{2}U^2 + 2\Phi\right) + \mathcal{O}^{-6}, \quad \tilde{n}^i = \frac{1}{c^3}\frac{1}{a}P^i + \mathcal{O}^{-5},\\ &\tilde{n}_0 = -1 + \frac{1}{c^2}U - \frac{1}{c^4}\left(\frac{1}{2}U^2 - 2\Phi\right) + \mathcal{O}^{-6}, \quad \tilde{n}_i \equiv 0. \qquad (33)\end{aligned}$$

By setting $\tilde{u}_i \equiv 0$ in Eq. (13) we also recover the normal vector, thus

$$v_i = \frac{1}{c^2}P_i + LT^{-1}\mathcal{O}^{-4}, \tag{34}$$

where v_i is, say, the velocity of the normal frame vector. The projection tensor based on $\tilde{n}^a$ becomes

$$\tilde{h}_{ij} = \tilde{g}_{ij}, \quad \tilde{h}_{0i} = \tilde{g}_{0i}, \quad \tilde{h}_{00} = \mathcal{O}^{-6}, \quad \tilde{h}^i_j = \delta^i_j, \quad \tilde{h}^i_0 = -\frac{1}{c^3}\frac{1}{a}P^i + \mathcal{O}^{-5}, \quad \tilde{h}^0_i = 0 = \tilde{h}^0_0. \tag{35}$$

Kinematic quantities based on $\tilde{n}^a$ become

$$\begin{aligned}\tilde{\theta} &= \frac{1}{c}3\frac{\dot{a}}{a} + \frac{1}{c^3}\left(3\dot{V} + 3\frac{\dot{a}}{a}U + \frac{1}{a}P^i{}_{|i}\right) + L^{-1}\mathcal{O}^{-5}, \qquad (36)\\ \tilde{\sigma}_{ij} &= \frac{1}{c^3}a\left(P_{(i|j)} - \frac{1}{3}P^k{}_{|k}\gamma_{ij}\right) + L^{-1}\mathcal{O}^{-5}, \qquad (37)\\ \tilde{\omega}_{ij} &= 0, \qquad (38)\\ \tilde{a}_i &= -\frac{1}{c^2}U_{,i} - \frac{1}{c^4}2\Phi_{,i} + L^{-1}\mathcal{O}^{-6}. \qquad (39)\end{aligned}$$

$\tilde{\sigma}_{0c}$ follows from $\tilde{\sigma}_{ac}\tilde{n}^c \equiv 0$, thus $\tilde{\sigma}_{0i} \sim L^{-1}\mathcal{O}^{-6}$ and $\tilde{\sigma}_{00} \sim L^{-1}\mathcal{O}^{-9}$; we have $\tilde{\omega}_{ab} = 0$. Similarly, $\tilde{a}_0$ follows from $\tilde{a}_c\tilde{n}^c \equiv 0$, thus $\tilde{a}_0 \sim L^{-1}\mathcal{O}^{-5}$.

The electric and the magnetic parts of Weyl tensor based on $\tilde{n}^a$ give

$$\tilde{E}^i_j = -\tilde{C}^i{}_{00j} = -\frac{1}{c^2}\frac{1}{a^2}\left[\frac{1}{2}(U+V)^{|i}{}_j - \frac{1}{6}\Delta(U+V)\delta^i_j\right] + L^{-2}\mathcal{O}^{-4}, \tag{40}$$

$$\tilde{H}^i_j = \frac{1}{2}\tilde{n}_0\tilde{n}_0\tilde{\eta}^{0ikl}\tilde{C}^0{}_{jkl} = \frac{1}{c^3}\frac{1}{2a^3}\eta^{ikl}\left[\frac{1}{2}\left(P^m{}_{|ml} - \Delta P_l + 2KP_l\right)\gamma_{jk} + P_{l|kj}\right] + L^{-2}\mathcal{O}^{-5}. \tag{41}$$

Thus, $\tilde{E}^i_j$ is the same in both frames to the 1PN order; compared with Eq. (32) $\tilde{H}^i_j$ differs. The nonvanishing K term which causes $\tilde{H}_{ab}$ to be not tracefree to $L^{-2}\mathcal{O}^{-3}$ can be ignored because the K term is of the $\mathcal{O}^{-2}$ order.

E. ADM quantities

In the Arnowitt-Deser-Misner (ADM) notation the metric and fluid quantities are [21]

$$\tilde{g}_{00} \equiv -N^2 + N^iN_i, \quad \tilde{g}_{0i} \equiv N_i, \quad \tilde{g}_{ij} \equiv h_{ij}, \tag{42}$$

$$\tilde{n}_0 \equiv -N, \quad \tilde{n}_i \equiv 0, \quad \tilde{n}^0 = N^{-1}, \quad \tilde{n}^i = -N^{-1}N^i, \tag{43}$$

$$E \equiv \tilde{n}_a\tilde{n}_b\tilde{T}^{ab}, \quad J_i \equiv -\tilde{n}_b\tilde{T}^b_i, \quad S_{ij} \equiv \tilde{T}_{ij}, \quad S \equiv h^{ij}S_{ij}, \quad \bar{S}_{ij} \equiv S_{ij} - \frac{1}{3}h_{ij}S, \tag{44}$$

where N_i, J_i and S_{ij} are based on h_{ij} as the metric, and h^{ij} is the inverse metric of h_{ij}. The extrinsic curvature is

$$K_{ij} \equiv \frac{1}{2N}\left(N_{i:j} + N_{j:i} - h_{ij,0}\right), \quad \bar{K} \equiv h^{ij}K_{ij}, \quad \bar{K}_{ij} \equiv K_{ij} - \frac{1}{3}h_{ij}\bar{K}, \tag{45}$$

where indices of K_{ij} are based on h_{ij}; a colon ':' denotes the covariant derivative based on h_{ij} with $\Gamma^{(h)i}{}_{jk} \equiv \frac{1}{2}h^{il}\left(h_{jl,k} + h_{lk,j} - h_{jk,l}\right)$. The intrinsic curvatures are based on the metric h_{ij}

$$R^{(h)i}{}_{jkl} \equiv \Gamma^{(h)i}{}_{jl,k} - \Gamma^{(h)i}{}_{jk,l} + \Gamma^{(h)m}{}_{jl}\Gamma^{(h)i}{}_{km} - \Gamma^{(h)m}{}_{jk}\Gamma^{(h)i}{}_{lm},$$

$$R^{(h)}_{ij} \equiv R^{(h)k}{}_{ikj}, \quad R^{(h)} \equiv h^{ij}R^{(h)}_{ij}, \quad \bar{R}^{(h)}_{ij} \equiv R^{(h)}_{ij} - \frac{1}{3}h_{ij}R^{(h)}. \tag{46}$$

A complete set of ADM equations will be presented in Sec. III C.

To the 1PN order we have

$$N = 1 - \frac{1}{c^2}U + \frac{1}{c^4}\left(\frac{1}{2}U^2 - 2\Phi\right) + \mathcal{O}^{-6},$$

$$N^i = -\frac{1}{c^3}\frac{1}{a}P^i\left(1 - \frac{1}{c^2}U\right) + \mathcal{O}^{-7}, \quad N_i = -\frac{1}{c^3}aP_i\left[1 + \frac{1}{c^2}(2V - U)\right] + \mathcal{O}^{-7},$$

$$h_{ij} = a^2\left(1 + \frac{1}{c^2}2V\right)\gamma_{ij} + \mathcal{O}^{-4}, \quad h^{ij} = \frac{1}{a^2}\left(1 - \frac{1}{c^2}2V\right)\gamma^{ij} + \mathcal{O}^{-4}, \tag{47}$$

$$\Gamma^{(h)i}{}_{jk} = \Gamma^{(\gamma)i}{}_{jk} + \frac{1}{c^2}\left(V_{,k}\delta^i_j + V_{,j}\delta^i_k - V^{,i}\gamma_{jk}\right) + L^{-1}\mathcal{O}^{-4}, \tag{48}$$

$$R^{(h)}_{ij} = R^{(\gamma)}_{ij} - \frac{1}{c^2}\left(V_{,i|j} + \gamma_{ij}\Delta V\right) + L^{-2}\mathcal{O}^{-4}, \quad R^{(h)} = \frac{1}{a^2}\left[6K - \frac{1}{c^2}4(\Delta + 3K)V\right] + L^{-2}\mathcal{O}^{-4}, \tag{49}$$

$$K_{ij} = -\frac{1}{c}a\dot{a}\gamma_{ij} - \frac{1}{c^3}a^2\left\{\left[\dot{V} + \frac{\dot{a}}{a}(U + 2V)\right]\gamma_{ij} + \frac{1}{a}P_{(i|j)}\right\} + L^{-1}\mathcal{O}^{-5},$$

$$\bar{K} = -\frac{1}{c}3\frac{\dot{a}}{a} - \frac{1}{c^3}\left[3\left(\dot{V} + \frac{\dot{a}}{a}U\right) + \frac{1}{a}P^i{}_{|i}\right] + L^{-1}\mathcal{O}^{-5}, \tag{50}$$

$$E = \varrho c^2 \left\{ 1 + \frac{1}{c^2}\left(v^2 + \Pi\right) + \frac{1}{c^4}\left[v^2 \left(v^2 + 2U + 2V + \Pi + \frac{p}{\varrho}\right) + 2U\Pi - 2P_i v^i + \frac{1}{\varrho}\left(2Q_i v^i + \Pi_{ij} v^i v^j\right)\right] + \mathcal{O}^{-6} \right\},$$

$$J_i = a\varrho c^2 \left\{ \frac{1}{c} v_i + \frac{1}{c^3}\left[v_i \left(v^2 + U + 2V + \Pi + \frac{p}{\varrho}\right) - P_i + \frac{1}{\varrho}\left(Q_i + \Pi_{ij} v^j\right)\right] + \mathcal{O}^{-5} \right\},$$

$$S_{ij} = a^2 \varrho c^2 \left\{ \frac{1}{c^2}\left(v_i v_j + \frac{p}{\varrho}\gamma_{ij} + \frac{1}{\varrho}\Pi_{ij}\right) \right.$$
$$\left. + \frac{1}{c^4}\left[v_i v_j \left(v^2 + 2U + 4V + \Pi + \frac{p}{\varrho}\right) + 2\frac{p}{\varrho} V \gamma_{ij} - 2 v_{(i} P_{j)} + \frac{2}{\varrho} Q_{(i} v_{j)}\right] + \mathcal{O}^{-6} \right\},$$

$$S = \varrho c^2 \left\{ \frac{1}{c^2}\left(v^2 + 3\frac{p}{\varrho}\right) + \frac{1}{c^4}\left[v^2 \left(v^2 + 2U + 2V + \Pi + \frac{p}{\varrho}\right) - 2P_i v^i + \frac{1}{\varrho}\left(2Q_i v^i + \Pi_{ij} v^i v^j\right)\right] + \mathcal{O}^{-6} \right\}. \quad (51)$$

III. DERIVATIONS

A. Equations of motion

Using Eq. (20) the energy and momentum conservation equations give

$$0 = -\frac{1}{c}\tilde{T}^b_{0;b}$$
$$= \frac{1}{a^3}\left(a^3\sigma\right)^{\cdot} + \frac{1}{a}\left[\sigma v^i + \frac{1}{c^2}\left(Q^i + \Pi^i_j v^j\right)\right]_{|i} + \frac{1}{c^2}\varrho\left[\dot{V} + \frac{1}{a} v^i \left(V - U\right)_{,i} + \frac{\dot{a}}{a} v^2 - \frac{\dot{p}}{\varrho}\right] + \varrho \mathcal{T}^{-1}\mathcal{O}^{-4}, \quad (52)$$

$$0 = \frac{1}{a}\tilde{T}^b_{i;b}$$
$$= \frac{1}{a^4}\left\{ a^4 \left[\sigma v_i + \frac{1}{c^2}\left(Q_i + \Pi_{ij} v^j\right)\right]\right\}^{\cdot} + \frac{1}{a}\left\{ \sigma v_i v^j + \Pi^j_i + \frac{1}{c^2}\left[Q^j v_i + Q_i v^j - 2\left(U + V\right)\Pi^j_i\right]\right\}_{|j}$$
$$+ \frac{1}{a}\left(1 - \frac{1}{c^2} 2U\right) p_{,i} - \frac{1}{a}\left(\sigma U_{,i} + \frac{1}{c^2}\varrho v^2 V_{,i}\right)$$
$$+ \frac{1}{c^2}\varrho\left\{ v_i \left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\left(U + 3V\right) + \frac{2}{a}\left[\left(U + V\right)U_{,i} - \Phi_{,i}\right] - \frac{1}{a}\left(aP_i\right)^{\cdot} - \frac{2}{a} v^j P_{[i|j]} + \frac{1}{\varrho a}\left(U + 3V\right)_{,j}\Pi^j_i \right\}$$
$$+ \varrho L \mathcal{T}^{-2}\mathcal{O}^{-4}, \quad (53)$$

where

$$\sigma \equiv \varrho\left[1 + \frac{1}{c^2}\left(v^2 + 2V + \Pi + \frac{p}{\varrho}\right)\right]. \quad (54)$$

For $Q_i = 0 = \Pi_{ij}$, $V = U$, $a = 1$ and $\gamma_{ij} = \delta_{ij}$ these equations reduce to Eqs. (64), (67) in Chandrasekhar [5].

Equation (52) can be written in another form as

$$\frac{1}{a^3}\left(a^3\varrho^*\right)^{\cdot} + \frac{1}{a}\left(\varrho^* v^i\right)_{|i} + \frac{1}{c^2}\left[\frac{1}{a}\left(Q^i{}_{|i} + \Pi^i_j v^j{}_{|i}\right) + \varrho\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\Pi + \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\mathbf{v}\right)p\right] + \varrho\mathcal{T}^{-1}\mathcal{O}^{-4} = 0, \quad (55)$$

where

$$\varrho^* \equiv \varrho\frac{\sqrt{-\tilde{g}}}{a^3\sqrt{\gamma}}\tilde{u}^0 \equiv \varrho\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + 3V\right) + \mathcal{O}^{-4}\right], \quad (56)$$

with $a^3\sqrt{\gamma}$ the background part of $\sqrt{-\tilde{g}}$. This corresponds to Eq. (117) in Chandrasekhar [5]; see also Eq. (44) in [6]. The mass conservation (continuity) equation $0 = \left(\tilde{\varrho}\tilde{u}^c\right)_{;c} = \dot{\tilde{\tilde{\varrho}}} + \tilde{\theta}\tilde{\varrho}$ gives

$$
\begin{aligned}
0 = c\left(\tilde{\varrho}\tilde{u}^c\right)_{;c} &= c\frac{1}{\sqrt{-\tilde{g}}}\left(\varrho\sqrt{-\tilde{g}}\tilde{u}^c\right)_{,c} \\
&= \frac{1}{a^3}\left(a^3\varrho^*\right)^{\cdot} + \frac{1}{a}\left(\varrho^* v^i\right)_{|i} + \varrho\mathcal{T}^{-1}\mathcal{O}^{-4} \\
&= \frac{1}{a^3}\left(a^3\varrho\right)^{\cdot} + \frac{1}{a}\left(\varrho v^i\right)_{|i} + \frac{1}{c^2}\varrho\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\left(\frac{1}{2}v^2 + 3V\right) + \varrho\mathcal{T}^{-1}\mathcal{O}^{-4}. \qquad (57)
\end{aligned}
$$

Thus, if we *assume* the mass conservation, Eq. (55) gives

$$
\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\Pi + \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\mathbf{v}\right)\frac{p}{\varrho} = -\frac{1}{\varrho a}\left(Q^i{}_{|i} + \Pi^i_j v^j{}_{|i}\right) + \mathcal{T}^{-1}\mathcal{O}^{-2}. \qquad (58)
$$

The specific entropy is introduced as $\tilde{T}d\tilde{S} = d\left(\tilde{\Pi}/c^2\right) + \tilde{p}/c^2 d\left(1/\tilde{\varrho}\right)$, thus along the flow we have

$$
\tilde{T}\dot{\tilde{S}} = \frac{1}{c^2}\left[\dot{\tilde{\Pi}} + \tilde{p}\left(1/\tilde{\varrho}\right)^{\tilde{\cdot}}\right] = \frac{1}{c^3}\left[\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\tilde{\Pi} + \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\mathbf{v}\right)\frac{\tilde{p}}{\tilde{\varrho}} + \mathcal{T}^{-1}\mathcal{O}^{-2}\right]. \qquad (59)
$$

This shows that for an adiabatic fluid flow the LHS of Eq. (58) vanishes; this also makes the RHS to vanish, which is naturally required for an ideal fluid. According to Chandrasekhar [6]: "*the conservation of mass and the conservation of entropy are not independent requirements in the framework of general relativity.* And the reason for their *independence* in the Newtonian limit is that in this limit ("$c^2 \to \infty$") [our Eq. (57)] reduces simply to the equation of continuity."

In [22] Blanchet, et al. suggested to use the following new combination of variable instead of v_i

$$
v_i^* \equiv \frac{1}{c}\frac{\sqrt{-\tilde{g}}}{a^4\sqrt{\gamma}}\frac{1}{\varrho^*}\tilde{T}^0_i, \qquad (60)
$$

which becomes

$$
v_i^* = v_i + \frac{1}{c^2}\left[v_i\left(\frac{1}{2}v^2 + U + 2V + \Pi + \frac{p}{\varrho}\right) - P_i + \frac{1}{\varrho}\left(Q_i + \Pi_{ij}v^j\right)\right] + L\mathcal{T}^{-1}\mathcal{O}^{-4}. \qquad (61)
$$

Notice that v_i^* is directly related to the ADM flux vector in Eq. (51)

$$
J_i = a\varrho c v_i^*. \qquad (62)
$$

Equation $\tilde{T}^b_{i;b} = 0$ can be written as follows

$$
\begin{aligned}
\frac{1}{c}\left(\frac{\sqrt{-\tilde{g}}}{\sqrt{\gamma}}\tilde{T}^0_i\right)^{\cdot} + \left(\frac{\sqrt{-\tilde{g}}}{\sqrt{\gamma}}\tilde{T}^j_i\right)_{|j} &= \frac{1}{2}\frac{\sqrt{-\tilde{g}}}{\sqrt{\gamma}}\tilde{T}^{ab}\tilde{g}_{ab|i} \\
&= \frac{\sqrt{-\tilde{g}}}{\sqrt{\gamma}}\varrho\left\{U_{,i} + \frac{1}{c^2}\left[U_{,i}\left(v^2 + \Pi\right) + V_{,i}\left(v^2 + 3\frac{p}{\varrho}\right) - v^j P_{j|i} + 2\Phi_{,i}\right] + L^2\mathcal{T}^{-2}\mathcal{O}^{-4}\right\}, \qquad (63)
\end{aligned}
$$

and we have

$$
\frac{\sqrt{-\tilde{g}}}{\sqrt{\gamma}}\tilde{T}^j_i = a^3 v^j\varrho^* v_i^* + \frac{\sqrt{-\tilde{g}}}{\sqrt{\gamma}}\left[p\delta^j_i + \Pi^j_i + \frac{1}{c^2}\left(Q^j v_i - 2V\Pi^j_i - \Pi_{ik}v^j v^k\right) + L\mathcal{T}^{-1}\mathcal{O}^{-4}\right]. \qquad (64)
$$

Thus, Eq. (63) can be arranged in the following form

$$
\begin{aligned}
\frac{1}{a}\left(av_i^*\right)^{\cdot} + \frac{1}{a}v^*_{i|j}v^j &= \frac{1}{a}\left\{U_{,i} + \frac{1}{c^2}\left[U_{,i}\left(\frac{1}{2}v^2 - U + \Pi\right) + V_{,i}\left(v^2 + 3\frac{p}{\varrho}\right) - v^j P_{j|i} + 2\Phi_{,i}\right]\right\} \\
&\quad - \frac{1}{\varrho^* a}\left\{\left[1 + \frac{1}{c^2}\left(3V - U\right)\right]\left[p\delta^j_i + \Pi^j_i + \frac{1}{c^2}\left(Q^j v_i - 2V\Pi^j_i - \Pi_{ik}v^j v^k\right)\right]\right\}_{|j} \\
&\quad + \frac{1}{c^2}\frac{v_i^*}{\varrho^*}\left[\frac{1}{a}\left(Q^j + \Pi^j_k v^k\right)_{|j} + \varrho\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\Pi + \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\mathbf{v}\right)p\right] + L\mathcal{T}^{-2}\mathcal{O}^{-4}, \qquad (65)
\end{aligned}
$$

where the last line vanishes in an adiabatic ideal fluid.

To the $\mathcal{O}^{-0}$ order, Eqs. (52), (53) give the Newtonian mass and momentum conservation equations, respectively

$$\frac{1}{a^3}\left(a^3\varrho\right)^{\cdot}+\frac{1}{a}\nabla_i\left(\varrho v^i\right)=0, \tag{66}$$

$$\frac{1}{a}\left(av_i\right)^{\cdot}+\frac{1}{a}v^j\nabla_j v_i+\frac{1}{a\varrho}\left(\nabla_i p+\nabla_j\Pi^j_i\right)-\frac{1}{a}\nabla_i U=0. \tag{67}$$

Thus, we naturally have the Newtonian equations to 0PN order, e.g., see [23]. This can be compared with the situation in perturbation approach where the Newtonian correspondence can be achieved only after suitable choice of different gauges for different variables, thus being a non-trivial result even to the linear order in perturbation approach [3,24]. Equations (66), (67), as well as all the 1PN equations in this section, are fully nonlinear.

In the Friedmann background we set the PN variables U, Φ, P_i, V, v_i, Q_i, and Π_{ij} equal to zero, and set $\varrho=\varrho_b$, $\Pi=\Pi_b$, and $p=p_b$, with $\mu\equiv\varrho\left(c^2+\Pi\right)$ and $\sigma_b\equiv\varrho_b\left[1+\frac{1}{c^2}\left(\Pi_b+p_b/\varrho_b\right)\right]$. Equations (52), (57) give

$$\dot{\mu}_b+3\frac{\dot{a}}{a}\left(\mu_b+p_b\right)=0, \tag{68}$$

$$\dot{\varrho}_b+3\frac{\dot{a}}{a}\varrho_b=0. \tag{69}$$

By subtracting the background part, Eq. (52) becomes

$$\frac{1}{a^3}\left[a^3\left(\sigma-\sigma_b\right)\right]^{\cdot}+\frac{1}{a}\left[\sigma v^i+\frac{1}{c^2}\left(Q^i+\Pi^i_j v^j\right)\right]_{|i}+\frac{1}{c^2}\varrho\left[\dot{V}+\frac{1}{a}v^i\left(V-U\right)_{,i}+\frac{\dot{a}}{a}v^2-\frac{\dot{p}-\dot{p}_b}{\varrho}\right]+\varrho\mathcal{T}^{-1}\mathcal{O}^{-4}=0. \tag{70}$$

B. Einstein's equations

We take Einstein's equations in the form

$$\tilde{R}^a_b=\frac{8\pi G}{c^4}\left(\tilde{T}^a_b-\frac{1}{2}\tilde{T}\delta^a_b\right)+\Lambda\delta^a_b. \tag{71}$$

Dimensions are as follows

$$[\tilde{g}_{ab}]=1,\quad [\tilde{R}_{ab}]=[\tilde{R}^a_b]=[\tilde{R}^{ab}]=L^{-2},\quad [\tilde{T}_{ab}]=[\tilde{T}]=[\varrho c^2]=ML^{-1}\mathcal{T}^{-2},\quad [\Lambda]=L^{-2},\quad [G\varrho]=\mathcal{T}^{-2}, \tag{72}$$

where M indicates the dimension of mass. To 1PN order, Eqs. (8), (20) give

$$-\Lambda+\frac{1}{c^2}\left(3\frac{\ddot{a}}{a}+\frac{\Delta}{a^2}U+4\pi G\varrho\right)+\frac{1}{c^4}\Bigg\{3\ddot{V}+3\frac{\dot{a}}{a}\left(\dot{U}+2\dot{V}\right)+6\frac{\ddot{a}}{a}U$$
$$-\frac{1}{a^2}\left[U^{,i}\left(U-V\right)_{,i}+2V\Delta U-2\Delta\Phi-\left(aP^i{}_{|i}\right)^{\cdot}\right]+8\pi G\varrho\left(v^2+\frac{1}{2}\Pi+\frac{3}{2}\frac{p}{\varrho}\right)\Bigg\}=0, \tag{73}$$

$$\frac{1}{c^3}\left[2\left(\dot{V}+\frac{\dot{a}}{a}U\right)_{,i}+\frac{1}{2a}\left(P^j{}_{|ji}-\Delta P_i-2KP_i\right)-8\pi G\varrho a v_i\right]=0, \tag{74}$$

$$\left(\frac{2K}{a^2}-\Lambda\right)\delta^i_j+\frac{1}{c^2}\left[\left(\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}-\frac{\Delta+4K}{a^2}V-4\pi G\varrho\right)\delta^i_j+\frac{1}{a^2}\left(U-V\right)^{|i}{}_j\right]$$
$$=\frac{1}{c^4}8\pi G\varrho\left[v^iv_j+\frac{1}{2}\left(\Pi-\frac{p}{\varrho}\right)\delta^i_j+\frac{1}{\varrho}\Pi^i_j\right], \tag{75}$$

where Eqs. (73), (74), and (75) are $\tilde{R}^0_0$, $\tilde{R}^0_i$, and $\tilde{R}^i_j$ parts of Eq. (71), respectively.

We kept the $\mathcal{O}^{-4}$ term on the RHS of Eq. (75) in order to get the correct Friedmann background equation. To be consistent, we also need to keep the LHS to $\mathcal{O}^{-4}$ which will explicitly involve 2PN order variables. But we do not need such efforts for our background subtraction process because all the $\mathcal{O}^{-4}$ order terms involve PN order variables, which do not affect the Friedmann background. To get the Friedmann background we set U, Φ, P_i, V, v_i, Q_i, and Π_{ij} equal to zero, and set $\varrho = \varrho_b$, $\Pi = \Pi_b$, and $p = p_b$. Then, Eqs. (73), (75) give

$$\frac{3}{c^2}\left\{\frac{\ddot{a}}{a} + \frac{4\pi G}{3}\left[\varrho_b\left(1+\frac{\Pi_b}{c^2}\right) + \frac{3p_b}{c^2}\right] - \frac{\Lambda c^2}{3}\right\} = 0, \tag{76}$$

$$\frac{1}{c^2}\left\{\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - 4\pi G\left[\varrho_b\left(1+\frac{\Pi_b}{c^2}\right) - \frac{p_b}{c^2}\right] + \frac{2Kc^2}{a^2} - \Lambda c^2\right\}\delta^i_j = 0. \tag{77}$$

Thus, we have

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\varrho_b\left(1+\frac{\Pi_b}{c^2}\right) + \frac{3p_b}{c^2}\right] + \frac{\Lambda c^2}{3}, \tag{78}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\varrho_b\left(1+\frac{\Pi_b}{c^2}\right) - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}, \tag{79}$$

which are the Friedmann equations.

By *subtracting* the background equations, Eqs. (73)-(75) give

$$\frac{1}{c^2}\left[\frac{\Delta}{a^2}U + 4\pi G\left(\varrho - \varrho_b\right)\right] + \frac{1}{c^4}\left\{3\ddot{V} + 3\frac{\dot{a}}{a}\left(\dot{U} + 2\dot{V}\right) + 6\frac{\ddot{a}}{a}U - \frac{1}{a^2}\left[U^{,i}\left(U-V\right)_{,i} + 2V\Delta U - 2\Delta\Phi - \left(aP^i{}_{|i}\right)^{\cdot}\right]\right.$$
$$\left. + 8\pi G\left[\varrho v^2 + \frac{1}{2}\left(\varrho\Pi - \varrho_b\Pi_b\right) + \frac{3}{2}\left(p - p_b\right)\right]\right\} = 0, \tag{80}$$

$$\frac{1}{c^3}\left[2\left(\dot{V} + \frac{\dot{a}}{a}U\right)_{,i} + \frac{1}{2a}\left(P^j{}_{|ji} - \Delta P_i - 2KP_i\right) - 8\pi G\varrho a v_i\right] = 0, \tag{81}$$

$$\frac{1}{c^2}\left\{-\left[\frac{\Delta + 4K}{a^2}V + 4\pi G\left(\varrho - \varrho_b\right)\right]\delta^i_j + \frac{1}{a^2}\left(U-V\right)^{|i}{}_j\right\} = 0. \tag{82}$$

To $\mathcal{O}^{-2}$ order, Eq. (80) gives

$$\frac{\Delta}{a^2}U = -4\pi G\left(\varrho - \varrho_b\right), \tag{83}$$

which is Poisson's equation in Newton's gravity. Notice that in the Newtonian limit of our PN approximation the homogeneous part of density distribution is subtracted. This revised form of Poisson's equation is consistent with Newton's gravity, and in fact, is an improved form avoiding Jeans' swindle [25]. The decomposition of Eq. (82) into trace and tracefree parts gives

$$\frac{\Delta + 3K}{a^2}V = -4\pi G\left(\varrho - \varrho_b\right), \tag{84}$$

$$\left(U - V\right)^{|i}{}_j = KV\delta^i_j. \tag{85}$$

Thus, compared with Eq. (83), for $K = 0$ we have

$$V = U, \tag{86}$$

where we ignored any surface term S with $\Delta S \equiv 0$ and $S_{,i|j} \equiv 0$. Notice that we have $V = U$ even in the presence of anisotropic stress; this differs from the situation in the perturbation approach where U and V in the zero-shear gauge (setting $P^i{}_{|i} = 0$, see later) are different in the presence of the anisotropic stress [3]. For later use we take a divergence of Eq. (81) which gives

$$\frac{\Delta}{a^2}\left(\dot{V} + \frac{\dot{a}}{a}U\right) = 4\pi G\frac{1}{a}\left(\varrho v^i\right)_{|i} + \frac{K}{a^3}P^i{}_{|i}. \tag{87}$$

Here, we reached a point to address the issue related to the background curvature in our PN approach. By taking a divergence of Eq. (81) and using Eqs. (83), (84), and (66) we have

$$\frac{1}{c^3}\left(-6K\dot{V}-\frac{2K}{a}P^i{}_{|i}\right)=0, \qquad (88)$$

which is apparently inconsistent. This is because the K term in Eq. (79) is related to the c^{-2} higher order terms in the background Friedmann equation. We can check this point by expanding the equation to 2PN order; in that order we can find that the above term can be removed by subtracting the background equation. Thus, it looks like our PN approximation is properly applicable only for $K = 0$. In this sense, our K terms do not have significance because these are related to $\mathcal{O}^{-2}$ order higher terms through the background equation. At the moment, it is unclear whether this limitation of our 1PN approach to a nearly flat background is due to our subtraction process of the background equations (which spread over different PN orders), or intrinsic to the whole PN approximation. Meanwhile, recent observations favour the flat background world model with non-vanishing cosmological constant. We include the cosmological constant in our PN approximation which appears only in the background Friedmann equations. Ignoring K term we simply have $V = U$.

C. ADM approach

The ADM equations are [21,3,17]

$$R^{(h)}=\bar{K}^{ij}\bar{K}_{ij}-\frac{2}{3}\bar{K}^2+\frac{16\pi G}{c^4}E+2\Lambda, \qquad (89)$$

$$\bar{K}^j_{i:j}-\frac{2}{3}\bar{K}_{,i}=\frac{8\pi G}{c^4}J_i, \qquad (90)$$

$$\bar{K}_{,0}N^{-1}-\bar{K}_{,i}N^iN^{-1}+N^{:i}{}_iN^{-1}-\bar{K}^{ij}\bar{K}_{ij}-\frac{1}{3}K^2-\frac{4\pi G}{c^4}(E+S)+\Lambda=0, \qquad (91)$$

$$\bar{K}^i_{j,0}N^{-1}-\bar{K}^i_{j:k}N^kN^{-1}+\bar{K}_{jk}N^{i:k}N^{-1}-\bar{K}^i_kN^k{}_{:j}N^{-1}$$
$$=\bar{K}\bar{K}^i_j-\left(N^{:i}{}_j-\frac{1}{3}\delta^i_jN^{:k}{}_k\right)N^{-1}+\bar{R}^{(h)i}{}_j-\frac{8\pi G}{c^4}\bar{S}^i_j, \qquad (92)$$

$$E_{,0}N^{-1}-E_{,i}N^iN^{-1}-\bar{K}\left(E+\frac{1}{3}S\right)-\bar{S}^{ij}\bar{K}_{ij}+N^{-2}\left(N^2J^i\right)_{:i}=0, \qquad (93)$$

$$J_{i,0}N^{-1}-J_{i:j}N^jN^{-1}-J_jN^j{}_{:i}N^{-1}-\bar{K}J_i+EN_{,i}N^{-1}+S^j_{i:j}+S^j_iN_{,j}N^{-1}=0. \qquad (94)$$

Using the ADM quantities presented in Sec. II E we can show that Eqs. (89)-(94) give the same equations we already have derived from Einstein's equations and the energy and momentum conservations. Equation (89) gives Eqs. (79), (84). Equation (90) gives Eq. (81). Equation (91) gives Eq. (73). Equation (92) gives the tracefree part of Eq. (82). Equation (93), (94) give Eqs. (52), (53), respectively. The ADM equations are often used in the PN approach [22,28,13,26]. We present the ADM approach because the ADM equations show the fully nonlinear structure of Einstein's gravity in a form suitable for numerical treatment. In fact, Eq. (65) can be derived from Eq. (94) using the identification made in Eq. (62).

IV. 1PN EQUATIONS

A. Complete equations to 1PN order

Here we summarize the complete set of equations valid to 1PN order in the cosmological situation. The background variables a, ϱ_b, Π_b and p_b are provided by solving Eqs. (68), (69), (78), and (79)

$$\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}\left[\varrho_b\left(1+\frac{\Pi_b}{c^2}\right)+\frac{3p_b}{c^2}\right]+\frac{\Lambda c^2}{3}, \qquad (95)$$

$$\frac{\dot{a}^2}{a^2}=\frac{8\pi G}{3}\varrho_b\left(1+\frac{\Pi_b}{c^2}\right)-\frac{Kc^2}{a^2}+\frac{\Lambda c^2}{3}, \qquad (96)$$

$$\dot{\mu}_b+3\frac{\dot{a}}{a}(\mu_b+p_b)=0, \qquad (97)$$

$$\dot{\varrho}_b+3\frac{\dot{a}}{a}\varrho_b=0, \qquad (98)$$

where $\mu_b \equiv \varrho_b c^2 \left(1 + \Pi_b/c^2\right)$. In Eqs. (95)-(97) only two are independent, and from Eqs. (97), (98) we have $\varrho_b \dot{\Pi}_b + 3(\dot{a}/a) p_b = 0$.

To the Newtonian order we have Eqs. (66), (67), and (83)

$$\frac{1}{a^3}\left(a^3\varrho\right)^{\cdot} + \frac{1}{a}\nabla_i\left(\varrho v^i\right) = 0, \tag{99}$$

$$\frac{1}{a}\left(a v_i\right)^{\cdot} + \frac{1}{a}v^j\nabla_j v_i + \frac{1}{a\varrho}\left(\nabla_i p + \nabla_j \Pi^j_i\right) - \frac{1}{a}\nabla_i U = 0, \tag{100}$$

$$\frac{\Delta}{a^2}U + 4\pi G\left(\varrho - \varrho_b\right) = 0. \tag{101}$$

In the Newtonian case energy conservation and mass conservation provide an additional equation which is Eq. (58)

$$\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\Pi + \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\mathbf{v}\right)\frac{p}{\varrho} + \frac{1}{\varrho a}\left(Q^i{}_{|i} + \Pi^i_j v^j{}_{|i}\right) = 0. \tag{102}$$

In the non-expanding background this equation gives the well known energy conservation equation in the Newtonian theory, see for example Eq. (2.36) in [27]. The Newtonian order hydrodynamic equations are valid in the presence of general background curvature K.

In the following we set $K = 0$ to the 1PN order, because our PN expansion fails to apply in the presence of K term. Thus we have $V = U$. The energy and the momentum conservation equations valid to 1PN order are derived in Eqs. (70), (53), and these are

$$\frac{1}{a^3}\left(a^3\sigma\right)^{\cdot} + \frac{1}{a}\left[\sigma v^i + \frac{1}{c^2}\left(Q^i + \Pi^i_j v^j\right)\right]_{|i} + \frac{1}{c^2}\varrho\left(\dot{U} + \frac{\dot{a}}{a}v^2 - \frac{\dot{p}}{\varrho}\right) = 0, \tag{103}$$

$$\begin{aligned}
&\frac{1}{a^4}\left\{a^4\left[\sigma v_i + \frac{1}{c^2}\left(Q_i + \Pi_{ij}v^j\right)\right]\right\}^{\cdot} + \frac{1}{a}\left[\sigma v_i v^j + \Pi^j_i + \frac{1}{c^2}\left(Q^j v_i + Q_i v^j - 4U\Pi^j_i\right)\right]_{|j} \\
&\quad + \frac{1}{a}\left(1 - \frac{1}{c^2}2U\right)p_{,i} - \frac{1}{a}\left(\sigma + \frac{1}{c^2}\varrho v^2\right)U_{,i} \\
&\quad + \frac{1}{c^2}\varrho\left[4v_i\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)U + \frac{2}{a}\left(U^2 - \Phi\right)_{,i} - \frac{1}{a}\left(aP_i\right)^{\cdot} - \frac{2}{a}v^j P_{[i|j]} + \frac{4}{\varrho a}U_{,j}\Pi^j_i\right] = 0,
\end{aligned} \tag{104}$$

where

$$\sigma \equiv \varrho\left[1 + \frac{1}{c^2}\left(v^2 + 2U + \Pi + \frac{p}{\varrho}\right)\right]. \tag{105}$$

To the Newtonian order Eqs. (103), (104) for ϱ and v_i reduce to Eqs. (99), (100), respectively. From Eq. (55), (65) we can derive alternative forms

$$\frac{1}{a^3}\left(a^3\varrho^*\right)^{\cdot} + \frac{1}{a}\left(\varrho^* v^i\right)_{|i} = -\frac{1}{c^2}\left[\frac{1}{a}\left(Q^i{}_{|i} + \Pi^i_j v^j{}_{|i}\right) + \varrho\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\Pi + \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\mathbf{v}\right)p\right], \tag{106}$$

$$\begin{aligned}
&\frac{1}{a}\left(a v^*_i\right)^{\cdot} + \frac{1}{a}v^*_{i|j}v^j = \frac{1}{a}\left\{U_{,i} + \frac{1}{c^2}\left[\left(\frac{3}{2}v^2 - U + \Pi + 3\frac{p}{\varrho}\right)U_{,i} + 2\Phi_{,i} - v^j P_{j|i}\right]\right\} \\
&\quad - \frac{1}{\varrho^* a}\left\{\left(1 + \frac{1}{c^2}2U\right)\left[p\delta^j_i + \Pi^j_i + \frac{1}{c^2}\left(Q^j v_i - 2V\Pi^j_i - \Pi_{ik}v^j v^k\right)\right]\right\}_{|j} \\
&\quad + \frac{1}{c^2}\frac{v^*_i}{\varrho^*}\left[\frac{1}{a}\left(Q^j + \Pi^j_k v^k\right)_{|j} + \varrho\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\Pi + \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\mathbf{v}\right)p\right],
\end{aligned} \tag{107}$$

where

$$\begin{aligned}
&\varrho^* \equiv \varrho\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + 3U\right)\right], \\
&v^*_i \equiv v_i + \frac{1}{c^2}\left[\left(\frac{1}{2}v^2 + 3U + \Pi + \frac{p}{\varrho}\right)v_i - P_i + \frac{1}{\varrho}\left(Q_i + \Pi_{ij}v^j\right)\right].
\end{aligned} \tag{108}$$

The hydrodynamic and thermodynamic variables Π, p, Q_i and Π_{ij} should be provided by specifying the equations of state and the thermodynamic state of the system under consideration.

The metric variables U, Φ and P_i can be expressed in terms of the Newtonian variables ϱ, v_i, Π, p, Q_i and Π_{ij} by using Einstein's equations. Equations (83), (84) give

$$\frac{\Delta}{a^2}U = -4\pi G\left(\varrho - \varrho_b\right), \tag{109}$$

which already determines U. Equations (80), (81), and (87) give

$$\frac{\Delta}{a^2}U + 4\pi G\left(\varrho - \varrho_b\right) + \frac{1}{c^2}\Bigg\{\frac{1}{a^2}\left[2\Delta\Phi - 2U\Delta U + \left(aP^i{}_{|i}\right)^{\cdot}\right] + 3\ddot{U} + 9\frac{\dot{a}}{a}\dot{U} + 6\frac{\ddot{a}}{a}U$$
$$+8\pi G\left[\varrho v^2 + \frac{1}{2}\left(\varrho\Pi - \varrho_b\Pi_b\right) + \frac{3}{2}\left(p - p_b\right)\right]\Bigg\} = 0, \tag{110}$$

$$\frac{\Delta}{a^2}P_i = -16\pi G\varrho v_i + \frac{1}{a}\left(\frac{1}{a}P^j{}_{|j} + 4\dot{U} + 4\frac{\dot{a}}{a}U\right)_{,i}, \tag{111}$$

$$\frac{\Delta}{a^2}\dot{U} = 4\pi G\left[\frac{1}{a}\left(\varrho v^i\right)_{|i} + \frac{\dot{a}}{a}\left(\varrho - \varrho_b\right)\right]. \tag{112}$$

Notice that to 1PN order Einstein's equations do not involve the anisotropic stress or flux term. To 1PN order Eq. (110) determines Φ. Our conservation equations (103), (104) contain $\dot{U}$ terms. In order to handle these terms it was suggested in [28] that the Poisson-type equation in Eq. (112) provides better numerical accuracy. Notice that, whereas only the spatial gradient of the potential U appears in the Newtonian limit, we have bare U terms present in the 1PN order [29].

In order to handle these equations we have the freedom to take one temporal gauge condition. This corresponds to imposing a condition on $P^i{}_{|i}$ or Φ. Gauge related issues will be addressed in detail in Sec. V. There, we will show that all our variables in this section are spatially gauge-invariant, and are temporally gauge-ready.

B. Ideal fluid case

We consider an ideal fluid, i.e., $Q_i \equiv 0 \equiv \Pi_{ij}$, and assume the adiabatic condition Eq. (102) applies. Thus, the internal energy is determined by the energy conservation equation

$$\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\Pi + \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\mathbf{v}\right)\frac{p}{\varrho} = 0. \tag{113}$$

To the background order we have Eqs. (95)-(98). To the Newtonian order we have Eqs. (99)-(102). Equations (103), (104) become

$$\frac{1}{a^3}\left(a^3\sigma\right)^{\cdot} + \frac{1}{a}\left(\sigma v^i\right)_{|i} + \frac{1}{c^2}\varrho\left(\dot{U} + \frac{\dot{a}}{a}v^2 - \frac{\dot{p}}{\varrho}\right) = 0, \tag{114}$$

$$\frac{1}{a^4}\left(a^4\sigma v_i\right)^{\cdot} + \frac{1}{a}\left(\sigma v_i v^j\right)_{|j} - \frac{1}{a}\left(\sigma + \frac{1}{c^2}\varrho v^2\right)U_{,i} + \frac{1}{a}\left(1 - \frac{1}{c^2}2U\right)p_{,i}$$
$$+\frac{1}{c^2}\varrho\left[4v_i\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)U + \frac{2}{a}\left(U^2 - \Phi\right)_{,i} - \frac{1}{a}\left(aP_i\right)^{\cdot} - \frac{2}{a}v^jP_{[i|j]}\right] = 0, \tag{115}$$

where σ is given in Eq. (105). These equations can be written as

$$\frac{1}{a^3}\left(a^3\varrho\right)^{\cdot} + \frac{1}{a}\left(\varrho v^i\right)_{|i} + \frac{1}{c^2}\varrho\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\left(\frac{1}{2}v^2 + 3U\right) = 0, \tag{116}$$

$$\frac{1}{a}\left(av_i\right)^{\cdot} + \frac{1}{a}v_{i|j}v^j - \frac{1}{a}U_{,i} + \frac{1}{a}\frac{p_{,i}}{\varrho} + \frac{1}{c^2}\Bigg[-\frac{1}{a}\left(v^2 + 4U + \Pi + \frac{p}{\varrho}\right)\frac{p_{,i}}{\varrho}$$
$$+v_i\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\left(\frac{1}{2}v^2 + 3U + \Pi + \frac{p}{\varrho}\right) - \frac{1}{a}v^2U_{,i} + \frac{2}{a}\left(U^2 - \Phi\right)_{,i} - \frac{1}{a}\left(aP_i\right)^{\cdot} - \frac{2}{a}v^jP_{[i|j]}\Bigg] = 0. \tag{117}$$

Alternative forms follow from Eqs. (106), (107)

$$\frac{1}{a^3}\left(a^3\varrho^*\right)^{\cdot} + \frac{1}{a}\left(\varrho^* v^i\right)_{|i} = 0, \tag{118}$$

$$\frac{1}{a}\left(av_i^*\right)^{\cdot} + \frac{1}{a}v^*_{i|j}v^j = -\frac{1}{a}\left(1+\frac{1}{c^2}2U\right)\frac{p_{,i}}{\varrho^*} + \frac{1}{a}\left[1+\frac{1}{c^2}\left(\frac{3}{2}v^2 - U + \Pi + \frac{p}{\varrho}\right)\right]U_{,i} + \frac{1}{c^2}\frac{1}{a}\left(2\Phi_{,i} - v^j P_{j|i}\right), \tag{119}$$

where

$$\varrho^* \equiv \varrho\left[1+\frac{1}{c^2}\left(\frac{1}{2}v^2 + 3U\right)\right], \quad v_i^* \equiv v_i + \frac{1}{c^2}\left[\left(\frac{1}{2}v^2 + 3U + \Pi + \frac{p}{\varrho}\right)v_i - P_i\right]. \tag{120}$$

All these equivalent three sets of equations are written in Eulerian forms. Einstein's equations in (110)-(112) will provide the metric perturbation variables U, Φ and P_i in terms of the Newtonian fluid variables (ϱ, v^i, Π, and p), after taking one temporal gauge condition. The pressure term should be provided by an equation of state. In the case of a zero-pressure fluid with vanishing internal energy we can set $p = 0 = \Pi$. We can take any set of equations of motion depending on the mathematical convenience in numerical treatments.

V. GAUGE ISSUE

A. Gauge transformation

We consider the following transformation between two coordinates x^a and $\hat{x}^a$

$$\hat{x}^a \equiv x^a + \tilde{\xi}^a(x^e). \tag{121}$$

For a tensor quantity we use the tensor transformation property between x^a and $\hat{x}^a$ coordinates

$$\tilde{t}_{ab}(x^e) = \frac{\partial \hat{x}^c}{\partial x^a}\frac{\partial \hat{x}^d}{\partial x^b}\hat{\tilde{t}}_{cd}(\hat{x}^e). \tag{122}$$

Comparing tensor quantities at the same spacetime point, x^a, we can derive the gauge transformation property of a tensor quantity, see Eq. (226) in [17],

$$\begin{aligned}\hat{\tilde{t}}_{ab}(x^e) = &\tilde{t}_{ab}(x^e) - 2\tilde{t}_{c(a}\tilde{\xi}^c{}_{,b)} - \tilde{t}_{ab,c}\tilde{\xi}^c + 2\tilde{t}_{c(a}\tilde{\xi}^d{}_{,b)}\tilde{\xi}^c{}_{,d} + \tilde{t}_{cd}\tilde{\xi}^c{}_{,a}\tilde{\xi}^d{}_{,b}\\ &+\tilde{\xi}^d\left(2\tilde{\xi}^c{}_{,(a}\tilde{t}_{b)c,d} + 2\tilde{t}_{c(a}\tilde{\xi}^c{}_{,b)d} + \frac{1}{2}\tilde{t}_{ab,cd}\tilde{\xi}^c + \tilde{t}_{ab,c}\tilde{\xi}^c{}_{,d}\right).\end{aligned} \tag{123}$$

We considered $\tilde{\xi}^a$ to the second perturbational order which will turn out to be sufficient for the 1PN order. In the following we will consider the gauge transformation properties of the metric and the energy-momentum variables.

As the metric we consider the following more generalized form

$$\begin{aligned}\tilde{g}_{00} &\equiv -\left[1 - \frac{1}{c^2}2U + \frac{1}{c^4}\left(2U^2 - 4\Phi\right)\right] + \mathcal{O}^{-6},\\ \tilde{g}_{0i} &\equiv -\frac{1}{c^3}aP_i + \mathcal{O}^{-5},\\ \tilde{g}_{ij} &\equiv a^2\left[\left(1+\frac{1}{c^2}2V\right)\gamma_{ij} + \frac{1}{c^2}\left(2C_{,i|j} + 2C_{(i|j)} + 2C_{ij}\right)\right] + \mathcal{O}^{-4},\end{aligned} \tag{124}$$

where C_i is transverse ($C^i{}_{|i} \equiv 0$), and C_{ij} is transverse and tracefree ($C^j_{i|j} \equiv 0 \equiv C^i_i$); indices of C_i and C_{ij} are based on γ_{ij}. In Eq. (124) we introduced 10 independent metric components: U and Φ together (1-component), P_i (3-components), V (1-component), C (1-component), C_i (2-components), and C_{ij} (2-components). It is known that gravitational waves show up in the 2.5PN order [8]. Thus, we ignore the transverse and tracefree part, i.e., set $C_{ij} \equiv 0$ to 1PN order.

We wish to keep the metric in the form of Eq. (124) in *any* coordinate system. Thus, we take the transformation variable $\tilde{\xi}^a$ to be a PN-order quantity. We consider coordinate transformations which satisfy

$$\tilde{\xi}^0 \equiv \frac{1}{c}\xi^{(2)0} + \frac{1}{c^3}\xi^{(4)0} + \dots, \quad \tilde{\xi}^i \equiv \frac{1}{c^2}\frac{1}{a}\xi^{(2)i} + \dots, \tag{125}$$

where the indices of the $\xi^{(2)i}$'s are based on γ_{ij}.

The gauge transformation of Eq. (123) applied to the metric gives

$$\hat{U} = U + \dot{\xi}^{(2)0}, \tag{126}$$

$$\hat{\Phi} = \Phi + \frac{1}{2}\dot{\xi}^{(4)0} - \frac{1}{2}\dot{U}\xi^{(2)0} - \frac{1}{2a}U_{,i}\xi^{(2)i} - \frac{1}{2}\xi^{(2)0}\ddot{\xi}^{(2)0} - \frac{1}{4}\dot{\xi}^{(2)0}\dot{\xi}^{(2)0} - \frac{1}{2}\left(\frac{1}{a}\xi^{(2)i}\xi^{(2)0}{}_{,i}\right)^{\cdot}, \tag{127}$$

$$\xi^{(2)0}{}_{,i} = 0, \tag{128}$$

$$\hat{P}_i = P_i - \frac{1}{a}\xi^{(4)0}{}_{,i} + a\left(\frac{1}{a}\xi^{(2)}_i\right)^{\cdot} + \frac{2}{a}\left(U + \dot{\xi}^{(2)0}\right)\xi^{(2)0}{}_{,i} + \frac{1}{a}\dot{\xi}^{(2)0}{}_{,i}\xi^{(2)0} + \frac{1}{a^2}\left(\xi^{(2)j}\xi^{(2)0}{}_{,j}\right)_{,i}, \tag{129}$$

$$(\Delta + 3K)\,\Delta\hat{C} = (\Delta + 3K)\left(\Delta C - \frac{1}{a}\xi^{(2)i}{}_{|i}\right) + \frac{\Delta}{4a^2}\left(\xi^{(2)0,i}\xi^{(2)0}{}_{,i}\right) - \frac{3}{4a^2}\left(\xi^{(2)0}{}_{,i}\xi^{(2)0}{}_{,j}\right)^{|ij}, \tag{130}$$

$$(\Delta + 2K)\,\hat{C}_i = (\Delta + 2K)\left(C_i - \frac{1}{a}\xi^{(2)}_i\right) - \frac{1}{3a}\xi^{(2)j}{}_{|ji} - \frac{1}{a^2}\left(\xi^{(2)0}{}_{,i}\xi^{(2)0}{}_{,j}\right)^{|j}$$
$$+ \frac{1}{a}\nabla_i\Delta^{-1}\left\{\frac{4}{3}(\Delta + 3K)\,\xi^{(2)j}{}_{|j} + \frac{1}{a}\left[\left(\xi^{(2)0}{}_{,k}\xi^{(2)0}{}_{,j}\right)^{|kj}\right]\right\}, \tag{131}$$

$$(\Delta + 3K)\,\hat{V} = (\Delta + 3K)\left(V - \frac{\dot{a}}{a}\xi^{(2)0}\right) - \frac{\Delta + 2K}{4a^2}\left(\xi^{(2)0,i}\xi^{(2)0}{}_{,i}\right) + \frac{1}{4a^2}\left(\xi^{(2)0}{}_{,i}\xi^{(2)0}{}_{,j}\right)^{|ij}. \tag{132}$$

Equations (126), (127) follow from the transformation of $\tilde{g}_{00}$; Eqs. (128), (129) follow from $\tilde{g}_{0i}$; Eqs. (130)-(132) follow from $\tilde{g}_{ij}$.

Equation (128) shows that $\xi^{(2)0}$ is spatially constant,

$$\xi^{(2)0} = \xi^{(2)0}(t). \tag{133}$$

After this, Eq. (130) gives

$$\Delta\hat{C} = \Delta C - \frac{1}{a}\xi^{(2)i}{}_{|i}. \tag{134}$$

Thus, by choosing $C \equiv 0$ ($\Delta C \equiv 0$ is enough) as the gauge condition (this implies that we set $C = 0$ to be valid in any coordinate), we have

$$\xi^{(2)i}{}_{|i} = 0. \tag{135}$$

Imposing the conditions in Eqs. (133), (135), Eq. (131) gives

$$\hat{C}_i = C_i - \frac{1}{a}\xi^{(2)}_i. \tag{136}$$

Thus, by choosing $C_i \equiv 0$ to be valid in any coordinate, i.e., choosing $C_i \equiv 0$ as the gauge condition, we have

$$\xi^{(2)}_i = 0. \tag{137}$$

Under the conditions in Eqs. (133), (137) the remaining metric perturbation variables transform as

$$\hat{U} = U + \dot{\xi}^{(2)0}, \quad \hat{V} = V - \frac{\dot{a}}{a}\xi^{(2)0}, \quad \hat{\Phi} = \Phi + \frac{1}{2}\dot{\xi}^{(4)0} - \frac{1}{2}\dot{U}\xi^{(2)0} - \frac{1}{2}\xi^{(2)0}\ddot{\xi}^{(2)0} - \frac{1}{4}\dot{\xi}^{(2)0}\dot{\xi}^{(2)0},$$
$$\hat{P}_i = P_i - \frac{1}{a}\xi^{(4)0}{}_{,i}. \tag{138}$$

Notice that for non-vanishing $\xi^{(2)0}$, U and V transform differently under the gauge transformation even in the flat cosmological background. Since the spatially constant $\xi^{(2)0}$ can be absorbed by a global redefinition of the time coordinate, without losing generality we can set $\xi^{(2)0}$ equal to zero

$$\xi^{(2)0} \equiv 0, \tag{139}$$

thus allowing $V = U$ in any coordinate. If we set $\xi^{(2)0} = 0$ but do not take the spatial gauge which fixes $\xi^{(2)}_i$, from Eqs. (126)-(132), the metric variables transform as

$$\hat{U} = U, \tag{140}$$

$$\hat{\Phi} = \Phi + \frac{1}{2}\dot{\xi}^{(4)0} - \frac{1}{2a}U_{,i}\xi^{(2)i}, \tag{141}$$

$$\hat{P}_i = P_i - \frac{1}{a}\xi^{(4)0}{}_{,i} + a\left(\frac{1}{a}\xi^{(2)}_i\right)^{\cdot}, \tag{142}$$

$$\Delta\hat{C} = \Delta C - \frac{1}{a}\xi^{(2)i}{}_{|i}, \tag{143}$$

$$(\Delta + 2K)\,\hat{C}_i = (\Delta + 2K)\left(C_i - \frac{1}{a}\xi^{(2)}_i\right) - \frac{1}{3a}\xi^{(2)j}{}_{|ji} + \frac{4}{3a}\nabla_i\Delta^{-1}(\Delta + 3K)\,\xi^{(2)j}{}_{|j}, \tag{144}$$

$$\hat{V} = V. \tag{145}$$

Taking the spatial gauge conditions $C = 0 = C_i$ we have $\xi^{(2)}_i = 0 = \xi^{(2)0}$, and the remaining metric perturbation variables transform as

$$\hat{U} = U, \quad \hat{V} = V, \quad \hat{\Phi} = \Phi + \frac{1}{2}\dot{\xi}^{(4)0}, \quad \hat{P}_i = P_i - \frac{1}{a}\xi^{(4)0}{}_{,i}. \tag{146}$$

In the perturbation analysis we call $C \equiv 0 \equiv C_i$ the spatial C-gauge [17]. Apparently, under these gauge conditions the spatial gauge transformation function to 1PN order, $\xi^{(2)}_i$, is fixed completely, i.e., $\xi^{(2)}_i = 0$. By taking such gauge conditions, the only remaining gauge transformation function to 1PN order is $\xi^{(4)0}$. This temporal gauge transformation function affects only Φ and P_i. If we take $\Phi \equiv 0$ as the temporal gauge condition, we have $\dot{\xi}^{(4)0} = 0$, thus $\xi^{(4)0}$ does not vanish even after imposing the temporal gauge condition and has general dependence on spatial coordinate $\xi^{(4)0}(\mathbf{x})$. Thus, in this gauge, even after imposing the temporal gauge condition the temporal gauge mode is not fixed completely. Whereas, if we take $P^i{}_{|i} \equiv 0$ as the temporal gauge condition, we have $\xi^{(4)0} = 0$, thus the temporal gauge condition is fixed completely. In fact, from the gauge transformation properties of Φ and P_i we can make the following combinations

$$\Phi_{,i} + \frac{1}{2}(aP_i)^{\cdot}, \quad \Delta\Phi + \frac{1}{2}\left(aP^i{}_{|i}\right)^{\cdot}, \tag{147}$$

which are invariant under the temporal gauge transformation, i.e., temporally gauge-invariant. Meanwhile, U and V are already temporally gauge-invariant. Since our spatial C-gauge also has removed the spatial gauge transformation function completely, we can correspond each remaining variable to a unique (spatially and temporally) gauge-invariant combination. Thus, in this sense, we can equivalently regard all our remaining variables as gauge-invariant ones. For example, for $K = 0$, from Eqs. (140)-(145) we can show

$$\hat{P}_i + a\left(\hat{C}_i + \hat{C}_{,i}\right)^{\cdot} = P_i + a\,(C_i + C_{,i})^{\cdot} - \frac{1}{a}\xi^{(4)0}{}_{,i}. \tag{148}$$

Thus, $P_i + a\,(C_i + C_{,i})^{\cdot}$ is spatially gauge-invariant and becomes P_i under the spatial C-gauge. This implies that P_i under the spatial C-gauge is equivalent to a unique gauge-invariant combination $P_i + a\,(C_i + C_{,i})^{\cdot}$. Similarly, in the case of the temporal gauge, from Eq. (147) we can show that $\Delta\Phi$ under the $P^i{}_{|i} = 0$ gauge is the same as a unique gauge-invariant combination $\Delta\Phi + \frac{1}{2}(aP^i{}_{|i})^{\cdot}$. In this sense, under gauge conditions which fix the gauge mode completely, the remaining variables can be regarded as equivalently gauge-invariant ones with corresponding gauge-invariant variables.

In Eq. (1) we began our 1PN analysis by choosing the spatial C-gauge

$$C \equiv 0 \equiv C_i. \tag{149}$$

By examining Eqs. (140)-(145) we notice that the spatial C-gauge is most economic in fixing the spatial gauge mode completely without due alternative. In this sense the spatial C-gauge can be regarded as a unique choice and we do not lose any mathematical convenience by taking this spatial gauge condition. In the literature these conditions are often expressed in the following forms. The spatial component of the harmonic gauge condition sets the 1PN part of

$$\tilde{g}^{ab}\tilde{\Gamma}^i_{ab} = -\frac{1}{\sqrt{-\tilde{g}}}\left(\sqrt{-\tilde{g}}\tilde{g}^{ic}\right)_{,c} = \frac{1}{c^2}\frac{\Delta}{a^2}\left(C^{,i} + C^i\right) + L^{-1}\mathcal{O}^{-4}, \tag{150}$$

equal to zero. We can also set the 1PN part of

$$\tilde{g}^{jk}\left(\tilde{g}_{ij,k} - \frac{1}{3}\tilde{g}_{jk,i}\right) = \frac{1}{c^2}\Delta\left(\frac{4}{3}C_{,i} + C_i\right) + L^{-1}\mathcal{O}^{-4}, \tag{151}$$

equal to zero; notice that in Eqs. (150), (151) we assumed $K = 0$. In either case we arrive at Eq. (149). We have shown that under the conditions in Eq. (149), the spatial gauge transformation is fixed completely without losing any generality or convenience. Whereas, we have not chosen the temporal gauge condition which can be best achieved by imposing a condition on $P^i{}_{|i}$. We call this the gauge-ready strategy [18,17]. It is convenient to choose this remaining temporal gauge condition depending on the problem we encounter; or to try many different ones in order to find out the best suitable one or ones.

Now, let us consider the gauge transformation property of the energy-momentum tensor. We set $\xi^{(2)0} \equiv 0$. The gauge transformation property of Eq. (123) applied to the energy-momentum tensor in Eq. (20) gives

$$\hat{\varrho} = \varrho + \frac{1}{c^2}\varrho^{(2)}, \tag{152}$$

$$\hat{\Pi} = \Pi - \frac{1}{a}\frac{\varrho_{,i}}{\varrho}\xi^{(2)i} - \frac{\varrho^{(2)}}{\varrho}, \tag{153}$$

$$\hat{v}_i = v_i + \frac{1}{c^2}v_i^{(2)}, \tag{154}$$

$$\hat{Q}_i = Q_i - \varrho\left[v_i^{(2)} - a\left(\frac{1}{a}\xi_i^{(2)}\right)^{\cdot} + \frac{1}{a}v_j\xi^{(2)j}{}_{|i} + \frac{1}{a}v_{i|j}\xi^{(2)j}\right], \tag{155}$$

$$\hat{p} = p - \frac{1}{c^2}\frac{1}{a}\left(p_{,i}\xi^{(2)i} + \frac{2}{3}p\xi^{(2)i}{}_{|i} + \frac{2}{3}\Pi^i_j\xi^{(2)j}{}_{|i}\right), \tag{156}$$

$$\hat{\Pi}_{ij} = \Pi_{ij} - \frac{1}{c^2}\frac{1}{a}\left[2p\left(\xi^{(2)}_{(i|j)} - \frac{1}{3}\gamma_{ij}\xi^{(2)k}{}_{|k}\right) + \Pi_{ij|k}\xi^{(2)k} + 2\Pi_{k(i}\xi^{(2)k}{}_{|j)} - \frac{2}{3}\gamma_{ij}\Pi^k_l\xi^{(2)l}{}_{|k}\right]. \tag{157}$$

Equations (152), (153) follow from the transformation property of $\tilde{T}_{00}$; Eqs. (154), (155) follow from $\tilde{T}_{0i}$; Eqs. (156), (157) follow from $\tilde{T}_{ij}$. The transformation functions $\varrho^{(2)}$ and $v_i^{(2)}$ are not determined, see below.

If we take the spatial C-gauge, i.e., set $\xi_i^{(2)} \equiv 0$, we have

$$\hat{\varrho} = \varrho + \frac{1}{c^2}\varrho^{(2)}, \quad \hat{\Pi} = \Pi - \frac{\varrho^{(2)}}{\varrho}, \quad \hat{v}_i = v_i + \frac{1}{c^2}v_i^{(2)}, \quad \hat{Q}_i = Q_i - \varrho v_i^{(2)}, \quad \hat{p} = p + \mathcal{O}^{-4}, \quad \hat{\Pi}_{ij} = \Pi_{ij} + \mathcal{O}^{-4}. \tag{158}$$

Thus, we have

$$\hat{\varrho}\left(1 + \frac{1}{c^2}\hat{\Pi}\right) = \varrho\left(1 + \frac{1}{c^2}\Pi\right), \quad \hat{v}_i + \frac{1}{c^2}\frac{1}{\varrho}\hat{Q}_i = v_i + \frac{1}{c^2}\frac{1}{\varrho}Q_i. \tag{159}$$

Although the gauge transformation properties of ϱ and Π are not determined individually, the energy density $\mu \equiv \varrho\left(1 + \Pi/c^2\right)$ is gauge-invariant under our C-gauge. For vanishing Π we have $\varrho^{(2)} = 0$. Similarly, the gauge transformation properties of v_i and Q_i are not determined individually. For vanishing flux term in any coordinate, $Q_i = 0$, we have $v_i^{(2)} = 0$, thus $\hat{v}_i = v_i + \mathcal{O}^{-4}$. Thus, the gauge transformation property of v_i is determined only for vanishing flux term. From Eq. (108) we have

$$\hat{\varrho}^* = \varrho^* + \frac{1}{c^2}\varrho^{(2)}, \quad \hat{v}_i^* = v_i^* + \frac{1}{c^2}\left(\frac{1}{a}\xi^{(4)0}{}_{,i} - \frac{\varrho^{(2)}}{\varrho}v_i\right). \tag{160}$$

We can check that up to the 1PN order Einstein's equations and the energy and momentum conservation equations in Sec. IV are invariant under the gauge transformation. In the following we introduce several temporal gauge conditions each of which fixes the temporal gauge mode completely. In relativistic gravity, gauge conditions consist of four non-tensorial relations imposed on the metric tensor or the energy-momentum tensor. Our purpose is to employ the temporal gauge (slicing) condition to make the resulting equations simple for mathematical/numerical treatment. Depending on the situation we can also take alternative spatial gauge conditions for that purpose. In the following we *impose* the spatial C-gauge in Eq. (149), and set $K = 0$.

B. Chandrasekhar's gauge

Chandrasekhar's temporal gauge condition in his Eq. (24) of [5] corresponds to setting the 1PN part of

$$\tilde{g}^{ij}\left(\tilde{g}_{0i,j}-\frac{1}{2}\tilde{g}_{ij,0}\right)=-\frac{1}{c}3\frac{\dot{a}}{a}-\frac{1}{c^3}\left(\frac{1}{a}P^i{}_{|i}+3\dot{U}\right)+L^{-1}\mathcal{O}^{-5}, \tag{161}$$

equal to zero; in the literature this is often called the standard PN gauge [22]. We take

$$\frac{1}{a}P^i{}_{|i}+3\dot{U}+m\frac{\dot{a}}{a}U=0, \tag{162}$$

as Chandrasekhar's gauge. We used the freedom to add an arbitrary $m(\dot{a}/a)U$ term with m, a real number. In this case Eqs. (110), (111) give

$$\frac{\Delta}{a^2}P_i=-16\pi G\varrho v_i+\frac{1}{a}\left[\dot{U}-(m-4)\frac{\dot{a}}{a}U\right]_{,i}, \tag{163}$$

$$\frac{\Delta}{a^2}U+4\pi G(\varrho-\varrho_b)+\frac{1}{c^2}\left\{2\frac{\Delta}{a^2}\Phi-(m-3)\frac{\dot{a}}{a}\dot{U}+\left[(6-m)\frac{\ddot{a}}{a}-m\frac{\dot{a}^2}{a^2}\right]U\right.$$
$$\left.+8\pi G\left[\varrho v^2+\frac{1}{2}(\varrho\Pi-\varrho_b\Pi_b)+U(\varrho-\varrho_b)+\frac{3}{2}(p-p_b)\right]\right\}=0. \tag{164}$$

Therefore, U, P_i, and Φ are determined by Eqs. (109), (163), and (164). The variable $\dot{U}$ can be determined from Eq. (112). This completes our 1PN scheme based on Chandrasekhar's gauge. When we handle this complete set of 1PN equations numerically, we should monitor whether the chosen gauge condition is satisfied always; this could be used to control the numerical accuracy.

C. Uniform-expansion gauge

The expansion scalar of the normal-frame vector, $\tilde{\theta}\equiv\tilde{n}^c{}_{;c}$, is given in Eq. (36). It is the same as the trace of extrinsic curvature $\bar{K}$ with a minus sign, see Eq. (50). Taking the 1PN part of $\bar{K}$ equal to zero

$$\frac{1}{a}P^i{}_{|i}+3\dot{U}+3\frac{\dot{a}}{a}U\equiv 0, \tag{165}$$

can be naturally called the uniform-expansion gauge. In the literature it is often called the ADM gauge [22], or the maximal slicing condition in numerical relativity [30]. This condition corresponds to the $m=3$ case of Chandrasekhar's gauge in Eq. (162).

D. Transverse-shear gauge

The shear of the normal-frame vector is given in Eq. (37). Thus, the gauge condition

$$P^i{}_{|i}\equiv 0, \tag{166}$$

can be called the transverse-shear gauge. In this case Eqs. (110), (111) give

$$\frac{\Delta}{a^2}P_i=-16\pi G\varrho v_i+\frac{4}{a}\left(\dot{U}+\frac{\dot{a}}{a}U\right)_{,i}, \tag{167}$$

$$\frac{\Delta}{a^2}U+4\pi G(\varrho-\varrho_b)+\frac{1}{c^2}\left\{2\frac{\Delta}{a^2}\Phi+3\ddot{U}+9\frac{\dot{a}}{a}\dot{U}+6\frac{\ddot{a}}{a}U\right.$$
$$\left.+8\pi G\left[\varrho v^2+\frac{1}{2}(\varrho\Pi-\varrho_b\Pi_b)+U(\varrho-\varrho_b)+\frac{3}{2}(p-p_b)\right]\right\}=0. \tag{168}$$

Therefore, U, P_i, and Φ are determined by Eqs. (109), (167), and (168). $\dot{U}$ can be determined from Eq. (112). We still have $\ddot{U}$ which can be determined similarly by taking the time derivative of Eq. (112) and by using the background and the Newtonian equations in Eqs. (95)-(101).

E. Harmonic gauge

We have

$$\tilde{g}^{ab}\tilde{\Gamma}^{0}_{ab} = -\frac{1}{\sqrt{-\tilde{g}}}\left(\sqrt{-\tilde{g}}\tilde{g}^{0a}\right)_{,a} = \frac{1}{c}3\frac{\dot{a}}{a} + \frac{1}{c^3}\left(\frac{1}{a}P^i{}_{|i} + 4\dot{U} + 6\frac{\dot{a}}{a}U\right) + L^{-1}\mathcal{O}^{-5}. \quad (169)$$

The well known harmonic gauge condition sets the 1PN part of Eq. (169) equal to zero, thus we take

$$\frac{1}{a}P^i{}_{|i} + 4\dot{U} + m\frac{\dot{a}}{a}U \equiv 0, \quad (170)$$

where we used a freedom to add an arbitrary $m(\dot{a}/a)U$ term. Combined with Eq. (150) the full harmonic gauge condition can be expressed by setting the 1PN parts of $\tilde{g}^{ab}\tilde{\Gamma}^{c}_{ab}$ equal to zero. In the perturbation approach the harmonic gauge (often called de Donder gauge) is known to be a bad choice because the condition involves derivatives of the metric variables; this leads to an incomplete gauge fixing and consequently we would have to handle higher order differential equations which is unnecessary, see the Appendix in [31]. In this gauge Eqs. (110), (111) give

$$\frac{\Delta}{a^2}P_i = -16\pi G\varrho v_i - (m-4)\frac{\dot{a}}{a}\frac{1}{a}U_{,i}, \quad (171)$$

$$\frac{\Delta}{a^2}U + 4\pi G(\varrho - \varrho_b) + \frac{1}{c^2}\Bigg\{2\frac{\Delta}{a^2}\Phi - \ddot{U} - (m-1)\frac{\dot{a}}{a}\dot{U} + \left[(6-m)\frac{\ddot{a}}{a} - m\frac{\dot{a}^2}{a^2}\right]U + 8\pi G\left[\varrho v^2 + \frac{1}{2}(\varrho\Pi - \varrho_b\Pi_b) + U(\varrho - \varrho_b) + \frac{3}{2}(p - p_b)\right]\Bigg\} = 0. \quad (172)$$

Therefore, U, P_i, and Φ are determined by Eqs. (109), (171), and (172). The variable $\dot{U}$ can be determined from Eq. (112). We also have a $\ddot{U}$ term in Eq. (172) which might demand for a more involved numerical implementation as explained below Eq. (168). However, its presence makes the propagating nature of the 1PN order metric fluctuations apparent in this gauge condition. This time-delayed propagation of the gravitational field in the 1PN approximation can be compared with the action-at-a-distance nature of Poisson's equation in the Newtonian order in Eq. (172). We anticipate that the relativistic time-delayed propagation could lead to a secular (time cumulative) effect. This could be important even in the case in which the 1PN correction terms are small compared with the Newtonian terms; in the large-scale clustering regions we would expect $(v/c)^2 \sim GM/(Rc^2) \simeq 10^{-6} \sim 10^{-4}$, see Sec. VI. Using $\tilde{g}_{00} \equiv -1 + 2\mathbf{U}/c^2$ we have

$$\mathbf{U} \equiv U + \frac{1}{c^2}\left(-U^2 + 2\Phi\right) + c^2\mathcal{O}^{-4}, \quad (173)$$

thus

$$\begin{aligned}\Box\mathbf{U} \equiv \mathbf{U}^{;c}{}_{c} &= \frac{\Delta}{a^2}\mathbf{U} - \frac{1}{c^2}\left(\ddot{\mathbf{U}} + 3\frac{\dot{a}}{a}\dot{\mathbf{U}} + 2\mathbf{U}\frac{\Delta}{a^2}\mathbf{U}\right) + \mathcal{T}^{-2}\mathcal{O}^{-4} \\ &= \frac{\Delta}{a^2}U - \frac{1}{c^2}\left[\frac{\Delta}{a^2}\left(U^2 - 2\Phi\right) + \ddot{U} + 3\frac{\dot{a}}{a}\dot{U} + 2U\frac{\Delta}{a^2}U\right] + \mathcal{T}^{-2}\mathcal{O}^{-4}, \end{aligned} \quad (174)$$

and Eq. (172) can be written as

$$\Box\mathbf{U} + 4\pi G(\varrho - \varrho_b) + \frac{1}{c^2}\left\{\frac{\Delta}{a^2}\mathbf{U}^2 - (m-4)\frac{\dot{a}}{a}\dot{\mathbf{U}} + \left[(6-m)\frac{\ddot{a}}{a} - m\frac{\dot{a}^2}{a^2}\right]\mathbf{U} + 8\pi G\left[\varrho v^2 + \frac{1}{2}(\varrho\Pi - \varrho_b\Pi_b) + \frac{3}{2}(p - p_b)\right]\right\} = 0. \quad (175)$$

Thus, in the harmonic gauge condition the propagation speed of the gravitational potential $\mathbf{U}$ is the same as the speed of light c. (When we mention the propagation speed, we are considering the D'Alembertian part of the wave equation ignoring the nonlinear terms and background expansion. In this case we have $\Box\mathbf{U} = a^{-2}\Delta\mathbf{U} - c^{-2}\ddot{\mathbf{U}}$, and for $K = 0$, Δ becomes the ordinary Laplacian in flat space.) Apparently, the wave speed of the metric (potential) can take an arbitrary value depending on the temporal gauge condition we choose. For example, if we take

$$\frac{1}{a}P^i{}_{|i} + n\dot{U} + m\frac{\dot{a}}{a}U \equiv 0, \quad (176)$$

as the gauge condition, with n and m real numbers, Eqs. (110), (111) give

$$\frac{\Delta}{a^2}P_i = -16\pi G\varrho v_i - \frac{1}{a}\left[(n-4)\dot{U} + (m-4)\frac{\dot{a}}{a}U\right]_{,i}, \tag{177}$$

$$\frac{\Delta}{a^2}U + 4\pi G(\varrho - \varrho_b) + \frac{1}{c^2}\left\{2\frac{\Delta}{a^2}\Phi - (n-3)\ddot{U} - (2n+m-9)\frac{\dot{a}}{a}\dot{U} + \left[(6-m)\frac{\ddot{a}}{a} - m\frac{\dot{a}^2}{a^2}\right]U \right.$$
$$\left. + 8\pi G\left[\varrho v^2 + \frac{1}{2}(\varrho\Pi - \varrho_b\Pi_b) + U(\varrho - \varrho_b) + \frac{3}{2}(p - p_b)\right]\right\} = 0. \tag{178}$$

In this case, for $n \geq 3$, the speed of propagation corresponds to

$$\frac{c}{\sqrt{n-3}}, \tag{179}$$

which can take an *arbitrary* value depending on our choice of the value of n. It becomes c for $n = 4$ (e.g., the harmonic gauge), and infinity for $n = 3$ (e.g., Chandrasekhar's gauge and the uniform-expansion gauge). In the case of the transverse-shear gauge we have $n = 0$, thus Eq. (168) is no longer a wave equation.

F. Transformation between two gauges

Now, let us show how we can relate the equations and solutions known in one gauge condition to the ones in any other gauge condition. Our spatial C-gauge condition already fixed the gauge transformation function $\xi^{(2)}_i \equiv 0$, and we have $\xi^{(2)0} \equiv 0$. Under the remaining (temporal) gauge transformation

$$\hat{t} = t + \frac{1}{c^4}\xi^{(4)0}, \quad \hat{x}^i = x^i, \tag{180}$$

we have

$$\hat{\Phi} = \Phi + \frac{1}{2}\dot{\xi}^{(4)0}, \quad \hat{P}_i = P_i - \frac{1}{a}\xi^{(4)0}{}_{,i}, \tag{181}$$

and all the other variables are gauge-invariant. As an example, let us consider the two gauge conditions used in Sec. V D and Sec. V E. Let us assume that the x^a coordinate is the transverse-shear gauge in Sec. V D, and $\hat{x}^a$ is the harmonic gauge in Sec. V E. Thus, we have $P^i{}_{|i} \equiv 0$ in x^a, and $4\dot{U} + m\frac{\dot{a}}{a}U + a^{-1}P^i{}_{|i} \equiv 0$ in $\hat{x}^a$. From Eq. (181) we have

$$-\left(4\dot{U} + m\frac{\dot{a}}{a}U\right) = 0 - \frac{\Delta}{a^2}\xi^{(4)0}, \tag{182}$$

thus

$$\frac{\Delta}{a^2}\xi^{(4)0} = 4\dot{U} + m\frac{\dot{a}}{a}U. \tag{183}$$

Thus, when we transform the result known in the transverse-shear gauge to the harmonic gauge, we use Eq. (181) with $\xi^{(4)0}$ given in Eq. (183). All the other variables are invariant under the gauge transformation. We can show that under this transformation Eqs. (167), (168) become Eqs. (171), (172), respectively. In this way, if we know the solution in any gauge condition the rest of the solutions in other gauge conditions can be simply derived without solving the equations again. Thus, as in other gauge theories (like Maxwell's or Yang-Mills' theories) the gauge condition should be deployed to our advantage in handling the problem mathematically.

VI. DISCUSSION

Here we compare the PN approach with the perturbative one. The perturbation analysis is based on the perturbation expansion of the metric and energy-momentum variables in a given background. All perturbation variables are assumed to be small. In linear order perturbations we keep only the first-order deviations from the background [1–3], whereas in the weakly nonlinear perturbations we keep higher-order deviations to the desired order [17]. Contrary to the PN approach the perturbation analysis is applicable in the strong gravity regime and on all cosmological scales as long as the perturbations are linear or weakly nonlinear.

Meanwhile, in the PN approach, by assuming weak gravitational fields and slow motions, we try to provide the general relativistic correction terms for the Newtonian equations of motion. Thus, in the PN approach, in fact, we abandon the geometric spirit of general relativity and recover the concept of absolute space and absolute time. Although this could be regarded as a shortcoming of the PN approach, in this way it provides the relativistic effects in forms of the correction terms to the well known Newtonian equations, thus enabling us to use simpler conventional (numerical) treatment. We expand the metric and the energy-momentum variables in powers of v/c in a given background spacetime. In nearly virialized system we have $GM/(Rc^2) \sim (v/c)^2$ which is assumed to be small. Thus, no strong gravity situation is allowed and the results are valid inside horizon $\sqrt{GM/(Rc^2)} \sim R/(c/H) < 1$. In comparison to the perturbation approach, however, the resulting equations in the 1PN approximation can be regarded as fully nonlinear. As in the perturbative case, even in our cosmological PN approach we *assume* the presence of a Robertson-Walker cosmological background.

As we have summarized in the introduction, our studies of the weakly nonlinear regime of a zero-pressure cosmological medium showed that the Newtonian equations are quite successful even near the horizon scale where the fluctuations are supposed to be near linear stage. We have shown that to the second order in perturbations, except for the presence of gravitational waves, the relativistic equations coincide exactly with the Newtonian ones. The pure relativistic correction terms appearing in the third order are independent of the presence of the horizon and are small compared with the second-order terms by a factor $\delta T/T \sim 10^{-5}$, thus negligible.

In order to properly estimate the relativistic effects in the evolution of large-scale cosmic structures we have to implement our equations in a hydrodynamic cosmological numerical simulation. The PN correction terms are $(v/c)^2$ or $GM/(Rc^2)$ orders smaller than the Newtonian terms. In Newtonian numerical simulations the maximum large scale velocity field of a cluster flow reaches nearly $3000km/sec$ and the typical value for the velocity is about an order of magnitude smaller than this [32]. Thus, we may estimate the 1PN effect to be of the order $(v/c)^2 \simeq 10^{-6} \sim 10^{-4}$, thus quite small. We already mentioned that the PN approximation is applicable inside the horizon only. Considering the action-at-a-distance nature of the Newtonian gravity theory it is important to check the domain of validity of the Newtonian theory in the nonlinear evolution of cosmological structures. The PN approach provides a way to find out the relativistic effects in such a regime. We anticipate that the propagating nature of the gravitational field with finite speed in relativistic gravity theory, compared with the instantaneous propagation in Newton's theory, could lead to accumulative (or secular) relativistic effects. In order to estimate relativistic effects it would be appropriate to consider a single cold dark matter component as a zero-pressure fluid without internal energy. In such a case our equations in Sec. IV B with $p = 0 = \Pi$ and the metric variables U, Φ, and P_i presented in various gauge conditions in Sec. V B-V E would provide a complete set of equations expressed in various forms.

Large-scale structures are still near the linear regime, and due to the enhanced amplitude of the initial mass power spectrum in the small scale the gravitational evolution causes nonlinear regions to begin in the small scale and to propagate to larger scales [33]. The current belief is that in small scale structures, where the nonlinearity is important, the dynamical time-scale is much longer than the light travel time over this scale, thus the time dilation effect from relativistic gravity is not important. The 1PN order relativistic effects could be important in the tidal interactions among clusters of galaxies, where the dynamical time-scale could become substantial compared with the light crossing time of the scales involved [29].

We can perform numerical simulations based on any two different gauge conditions. The results should coincide after making a gauge transformation between the two gauges as in Sec. V F. Also, the two simulations should give identical result for any given gauge-invariant variable. These might provide a way to check the numerical accuracy of the simulations. Analyses based on different gauge conditions would lead to the different results; this is in the sense that a given variable evaluated in two different gauges are actually two different variables, unless the original variable is gauge-invariant. Even for the temporal gauge condition (after fixing the spatial C-gauge) there are infinitely many different gauge conditions available. Since each of the gauge conditions displayed in Sec. V B-V E fixes the temporal gauge mode completely (and the spatial gauge modes are already completely fixed by our spatial C-gauge), all remaining variables are equivalently gauge-invariant. Thus, apparently, the gauge-invariance does not guarantee us to associate the variables with physically measurable quantities. The identification of physically measurable quantities out of infinitely many gauge-invariant candidates is still an open issue which remains to be addressed. The gauge invariance assures that the value should not depend on the gauge we take and this fact can be used to check and control the numerical accuracy of the simulations.

In a related context, we have shown that the propagation speed of the gravitational potential depends on the temporal gauge condition we take, see Eq. (179). A similar situation occurs in classical electrodynamics. The propagation speeds of the electromagnetic scalar and vector potentials are the speed of light c in the Lorenz gauge, whereas that of scalar potential becomes infinite in the Coulomb gauge. In electromagnetism the issue is well resolved by showing that the electric and magnetic fields propagate with c independently of the adopted gauge condition [34]. In our 1PN approach, however, we have *not* been able to resolve the case. In the PN case this is related to identifying the physically relevant gauge condition out of infinitely many different gauge conditions available (which are all gauge-invariant), an issue we have described in the previous paragraph. One difference compared to electromagnetism is that the PN approach addresses fully nonlinear gravitational dynamics whereas electromagnetics concerns linear processes. We wish to address these important issues in a future occasion.

Just like the solar system tests of Einstein's gravity theory, the nonlinear evolution of the large-scale cosmic structure could provide another regime where the gravitational field is weak and the motions are slow so that the post-Newtonian approximation would be practically adequate to describe the ever-present relativistic effects using Newtonian-like equations. The problem is whether such effects are significant enough to be detected in future observations and numerical experiments. This important issue is left for future studies. We hope that our set of 1PN order equations and our strategy for using those equations would be useful for such studies in the cosmological context. In more realistic cosmological situations we have dust and cold dark matter which can be approximated by two zero-pressure ideal fluids. We can also include pressure and dissipation effects in the case of dust. The multi-component situation of cosmological 1PN hydrodynamics will be considered in later extensions of this work. Extending our formulation to higher order PN approximation is also an apparent next step which would be tedious but straightforward.

ACKNOWLEDGMENTS

We thank Prof. Dongsu Ryu and Dr. Juhan Kim for useful discussions from the perspective of possible numerical implementations. DP wishes to thank Prof. M. Pohl for his support of this project. HN was supported by grants No. R04-2003-10004-0 from the Basic Research Program of the Korea Science and Engineering Foundation. JH was supported by the Korea Research Foundation Grant No. 2003-015-C00253.

---

[1] E.M. Lifshitz, J. Phys. (USSR) **10**, 116 (1946); E.M. Lifshitz and I.M. Khalatnikov, Adv. Phys. **12**, 185 (1963).

[2] E.R. Harrison, Rev. Mod. Phys. **39**, 862 (1967); G.B. Field and L.C. Shepley, Astrophys. Space. Sci. **1**, 309 (1968); H. Nariai, Prog. Theor. Phys. **41**, 686 (1969); V.N. Lukash, Sov. Phys. JETP **52**, 807 (1980); P.J.E. Peebles, *The large-scale structure of the universe*, (Princeton Univ. Press, Princeton, 1980); Ya.B. Zel'dovich and I.D. Novikov, *Relativistic astrophysics, Vol 2, The structure and evolution of the universe*, (Univ. Chicago Press, Chicago, 1983); V.F. Mukhanov, Sov. Phys. JETP Lett. **41**, 493 (1985); J.M. Bardeen, in *Particle Physics and Cosmology*, edited by L. Fang and A. Zee (Gordon and Breach, London, 1988), p1.

[3] J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[4] V.A. Fock, J. Phys. (USSR) **1**, 81 (1939); Rev. Mod. Phys. **29**, 325 (1957); *The theory of space, time and gravitation* (Pergamon Press, Oxford, 1964).

[5] S. Chandrasekhar, Astrophys. J. **142**, 1488 (1965).

[6] S. Chandrasekhar, Astrophys. J. **158**, 45 (1969).

[7] S. Chandrasekhar and Y. Nutku, Astrophys. J. **158**, 55 (1969).

[8] S. Chandrasekhar and F.P. Esposito, Astrophys. J. **160**, 153 (1970).

[9] T. Damour, in *300 years of gravitation*, edited by S. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1987), 128p.

[10] H. Noh and J. Hwang, gr-qc/0412127 (2004); J. Hwang and H. Noh, gr-qc/0412128 (2004).

[11] J. Hwang and H. Noh, gr-qc/0412129 (2004).

[12] C.M. Will, *Theory and experiment in gravitational physics* (Cambridge Univ. Press, London, 1981).

[13] H. Asada and T. Futamase, Prog. Theor. Phys. Suppl. **128**, 123 (1997).

[14] V.A. Brumberg, *Essential relativistic celestial mechanics* (Adam Hilger, Briston, 1991).

[15] M.H. Soffel, *Relativity in astrometry, celestial mechanics and geodesy*, (Springer-Verlag, Berlin, 1989); M. Soffel, et al., Astron. J. **126**, 2687 (2003).

[16] J. Hwang and H. Noh, Phys. Rev. D **71**, 063536 (2005).

[17] H. Noh and J. Hwang, Phys. Rev. D **69**, 104011 (2004).

[18] J. Hwang, Astrophys. J. **375**, 443 (1991).

[19] P.J. Greenberg, Astrophys. J. **164**, 569 (1971).

[20] J. Ehlers, Proceedings of the mathematical-natural science of the Mainz academy of science and literature, Nr. **11**, 792 (1961), translated in Gen. Rel. Grav. **25**, 1225 (1993); G.F.R. Ellis, in *General relativity and cosmology, Proceedings of the international summer school of physics Enrico Fermi course 47*, edited by R. K. Sachs (Academic Press, New York, 1971), p104; in *Cargese Lectures in Physics*, edited by E. Schatzmann (Gorden and Breach, New York, 1973), p1.
[21] R. Arnowitt, S. Deser, and C.W. Misner, in *Gravitation: an introduction to current research*, edited by L. Witten (Wiley, New York, 1962) p. 227.
[22] L. Blanchet, T. Damour, and G. Schäfer, Mon. Not. R. Astron. Soc. **244**, 289 (1990).
[23] E. Bertschinger and A.J.S. Hamilton, Astrophys. J. **435**, 1 (1994); L. Kofman and D. Pogosyan, *ibid.* **442**, 30 (1995); G.F.R. Ellis and P.K.S. Dunsby, Astrophys. J. **479**, 97 (1997).
[24] J. Hwang and H. Noh, Gen. Rel. Grav. **31**, 1131 (1999).
[25] D.S. Lemons, Am. J. Phys. **56**, 502 (1988).
[26] M. Takada and T. Futamase, Mon. Not. R. Astron. Soc. **306**, 64 (1999).
[27] F.H. Shu, *The physics of astrophysics, Vol. II Gas dynamics* (University Science Books, California, 1992).
[28] M. Shibata and H. Asada, Prog. Theor. Phys. **94**, 11 (1995).
[29] D. Ryu, private communication (2005).
[30] L. Smarr and J.W. York, Phys. Rev. D **17**, 2529 (1978).
[31] J. Hwang, Astrophys. J. **415**, 486 (1993).
[32] H. Kang, D. Ryu, and T.W. Jones, Astrophys. J. **456**, 422 (1996).
[33] R.E. Smith, et al, Monthly. Not. R. Astron. Soc. **341**, 1311 (2003).
[34] J.D. Jackson, Am. J. Phys. **70**, 917 (2002).