

Demystifying autoparallels in alternative gravity

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Autoparallel curves along with geodesic curves can arise as trajectories of physical test bodies. We explicitly derive autoparallels as effective post-Riemannian geometric constructs, and at the same time we argue *against* postulating autoparallels as fundamental equations of motion for test bodies in alternative gravity theories.

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I. INTRODUCTION

The equations of motion of test bodies in relativistic gravity are tightly linked to the conservation laws of the theory [1–3]. The explicit derivation of these equations has been intertwined with the development of approximation schemes within relativistic gravity [4–6]. As it is well known, geodesic curves arise as trajectories of structureless test bodies in Riemannian spacetimes with the metric g_{ij} as the gravitational field potential, that determines the metric-compatible Christoffel connection $\tilde{\Gamma}_{ij}{}^k = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$.

In alternative gravity theories the set of the gravitational field variables is extended to $(g_{ij}, \Gamma_{ij}{}^k)$ where the connection and the metric are no longer compatible, so that the torsion $T_{ij}{}^k := \Gamma_{ij}{}^k - \Gamma_{ji}{}^k$ and the nonmetricity $Q_{kij} := -\nabla_k g_{ij} = -\partial_k g_{ij} + \Gamma_{ki}{}^l g_{lj} + \Gamma_{kj}{}^l g_{il}$ are non-trivial, in general. Autoparallel curves have been *postulated* on several occasions in the literature as candidates for the equations of motion of test bodies in alternative gravity theories. Such ad-hoc postulates, unsubstantiated by the conservation laws, usually lead to inconsistencies with the field equations. Consequently one should abstain from the practice of postulating equations of motion instead of deriving them [7, 8]. With this warning in mind, we here report on two *special* cases, in which autoparallel curves actually do emerge in theories with post-Riemannian spacetime structure.

Let us consider the dynamics of massive particles under the action of the gravitational g_{ij} and a scalar ϕ field. We demonstrate that it is possible to recast the latter into a geometric property of the underlying spacetime, and construct an effective torsion $T_{ij}{}^k$ and nonmetricity Q_{kij} from this scalar field. As a preliminary step, we recall that the deviation of the spacetime geometry from the Riemannian [9] one is described by the distortion tensor

which measures the difference of connections, $N_{ij}{}^k := \tilde{\Gamma}_{ij}{}^k - \Gamma_{ij}{}^k$. Explicitly:

$$N_{ij}{}^k = -\frac{1}{2}(T_{ij}{}^k - T_j{}^k{}_i + T^k{}_{ij}) + \frac{1}{2}(Q^k{}_{ij} - Q_{ij}{}^k - Q_{ji}{}^k). \quad (1)$$

II. EFFECTIVE TORSION FROM THE SCALAR FIELD

At first, we set the nonmetricity equal zero $Q_{kij} = 0$, and introduce the torsion tensor of the form

$$T_{ij}{}^k = \delta_i^k V_j - \delta_j^k V_i, \quad (2)$$

where the vector field

$$V_i = \xi \partial_i \phi \quad (3)$$

is the gradient of the scalar field ϕ with some arbitrary parameter ξ . By construction, the torsion (2) has only one irreducible part, namely, the trace $T_{ki}{}^k = 3V_i$.

Accordingly, we derive from (2) and (1) the distortion (which is equal to the contortion tensor in this case):

$$N_{ij}{}^k = g_{ij} V^k - \delta_i^k V_j. \quad (4)$$

Now we write down the autoparallel equation for a point particle with the velocity $u^i = \frac{dx^i}{ds}$:

$$\frac{Du^k}{ds} = \frac{d^2 x^k}{ds^2} + \Gamma_{ij}{}^k u^i u^j = 0. \quad (5)$$

Substituting $\Gamma_{ij}{}^k = \tilde{\Gamma}_{ij}{}^k - N_{ij}{}^k$ and using (4), we recast (5) into

$$\frac{d^2 x^k}{ds^2} + \tilde{\Gamma}_{ij}{}^k u^i u^j - (V^k - u^k u^i V_i) = 0. \quad (6)$$

With the help of (3), we finally rewrite (6) as

$$\frac{d^2 x^k}{ds^2} + \tilde{\Gamma}_{ij}{}^k u^i u^j = \xi (g^{ik} - u^k u^i) \partial_i \phi. \quad (7)$$

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III. EFFECTIVE NONMETRICITY FROM THE SCALAR FIELD

Another non-Riemannian interpretation of the equation of motion (7) can be achieved by setting the torsion equal zero $T_{ij}{}^k = 0$, and by using the nonmetricity of the form

$$Q_{kij} = -g_{ij}V_k + g_{k(i}V_{j)}. \quad (8)$$

As becomes clear from (8), such nonmetricity has two nontrivial irreducible parts: the 3rd and the 4th (Weyl vector). Substituting this into (1), we find the distortion tensor

$$N_{ij}{}^k = g_{ij}V^k - \delta_{(i}^k V_{j)}. \quad (9)$$

As a result, we discover that the autoparallel equation (5) again reads

$$\frac{d^2 x^k}{ds^2} + \tilde{\Gamma}_{ij}{}^k u^i u^j - (V^k - u^k u^i V_i) = 0. \quad (10)$$

Then, by identifying, as before, the vector field with the gradient of the scalar field (3), we finally obtain the same equation of motion (7).

IV. MORE ON AUTOPARALLELS

The possible role of autoparallels in alternative gravity has been the subject to several works [10–13]. As becomes clear from (1), the most general autoparallel equation may be written in terms of the metric, the nonmetricity, and the torsion and the well-known decompositions [9, 14] of these geometrical quantities may be used to study subcases.

$$\frac{d^2 x^k}{ds^2} + \left(\tilde{\Gamma}_{ij}{}^k - N_{ij}{}^k \right) u^i u^j = 0 \quad (11)$$

In view of the symmetry of the factor $u^i u^j$, all four irreducible parts of the nonmetricity tensor $Q_{ijk} = {}^{(1)}Q_{ijk} + {}^{(2)}Q_{ijk} + {}^{(3)}Q_{ijk} + {}^{(4)}Q_{ijk}$ and two irreducible pieces of the torsion tensor $T_{ijk} = {}^{(1)}T_{ijk} + {}^{(2)}T_{ijk}$ (the traceless tensor part, and the vector trace of the torsion) can contribute to (11), excluding the axial trace of the torsion. It is possible to develop a corresponding classification of the autoparallel curves, thereby generalizing the earlier findings of [15] from Riemann-Cartan spaces to metric-affine geometries. Here we have demonstrated the distinguished role of the vector parts of the torsion and the nonmetricity.

However, it should be stressed, as it has been shown for a very large class of gravitational theories [16], that the autoparallel equation does *not* emerge as the equation of motion.

One should carefully distinguish the physical and the mathematical aspects when analyzing the motion of test

particles. From the mathematical point of view, it is possible to postulate any trajectories in the post-Riemannian spacetimes, and autoparallel curves may look indeed like a reasonable choice which describes the “straightest” curve between any two points on a manifold. However, from the point of view of physics, we should explain the physical reason which forces a test particle to move along such a trajectory. In this sense, the following analogy could be helpful. Let us consider an electrically neutral but massive point particle on a Riemannian manifold, which moves, as one knows, along the Riemannian geodesic curves. This motion is a result of an action of the gravitational field on particle’s mass. Suppose, however, that in addition we switch an electric or magnetic (or, in general, an electromagnetic) field on. Does this change the motion of a neutral test particle? The answer is crystal clear: no. Such a particle will still move along the Riemannian geodesic trajectory, without feeling the presence of electric and magnetic fields, no matter how strong they are, because it does not carry an electric charge. The situation with the motion of a massive point particle in post-Riemannian geometries is completely analogous. Since the mass is the only physical property (= “gravitational charge”) of a particle, the latter plainly cannot feel any additional post-Riemannian geometrical fields, such as the torsion and the nonmetricity. Just like one needs to attach an electric charge to a test particle in order to force it to deviate from the geodesic motion under the action of an electromagnetic field, in order to force a particle to move along an autoparallel instead of a geodesic curve, one should attach to it an additional “charge” that interacts with the torsion and/or nonmetricity. Indeed, the corresponding analysis [7, 16] of the equations of motion in metric-affine gravity (MAG) reveals that one needs matter with microstructure (i.e., with spin, dilation and/or shear charge) to feel post-Riemannian parts of the gravitational field.

V. CONCLUSION

In order to demonstrate the physical relevance of the equation of motion above, let us consider the modified action of a structureless point particle

$$I = \int \varphi ds = \int \varphi(x) \sqrt{g_{ij}(x) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}} d\lambda. \quad (12)$$

Such a nonminimal coupling, where the function φ can depend arbitrary on the spacetime coordinates (either directly, or via the geometric invariants), can be viewed as a description of the motion of a body with a variable mass [17].

The variation with respect to the coordinates $x^k(s)$ of

the particle is straightforwardly computed

$$\begin{aligned}
\delta I &= \int ds \left[\varphi \left(\frac{1}{2} \delta g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} + g_{ij} \frac{dx^i}{ds} \frac{d\delta x^j}{ds} \right) + \delta\varphi \right] \\
&= \int ds \delta x^k \left[\frac{1}{2} \varphi (\partial_k g_{ij}) u^i u^j - \frac{d}{ds} (\varphi g_{ki}) u^i - \varphi g_{ki} \frac{du^i}{ds} + \partial_k \varphi \right] \\
&= - \int ds g_{ik} \delta x^k \left[\varphi \left(\frac{du^i}{ds} + \tilde{\Gamma}_{mn}{}^i u^m u^n \right) - \partial^i \varphi + \frac{d\varphi}{ds} u^i \right].
\end{aligned} \tag{13}$$

Accordingly, the minimal action principle $\delta I = 0$ yields the equation of motion

$$\frac{d^2 x^i}{ds^2} + \tilde{\Gamma}_{mn}{}^i u^m u^n = (g^{ij} - u^i u^j) \partial_j \log \varphi. \tag{14}$$

By setting $\varphi = e^{\xi\phi}$, we thus conclude that *formally* there exist two possibilities to recast the equation of motion (14) into an *autoparallel* in a non-Riemannian spacetime: with the effective torsion (2) or with the effective nonmetricity (8). In principle, one could even think of a combination of the two cases into a more general spacetime geometry with both torsion and nonmetricity.

It is instructive to compare the above results to the equations of motion of a spinless particle nonminimally coupled to the gravitational field, derived from first principles on the basis of the conservation laws [18–20]. Therein, by means of multipolar techniques it was shown, for a very large class of gravitational theories, that test bodies perform a non-geodesic motion with a “pressure”-like force. At the first sight, the resulting equations are equivalent to the autoparallel curve (5) found above. However, one should be clear about the fact that this similarity is *purely formal*. In contrast to the derivation given in [18–20], the effective torsion (2) and the nonmetricity (8) are *not* linked to any kind of gravitational field equations.

Alternative theories of gravity can be studied in the unified framework of metric-affine gravity [14]. The lat-

ter is based on gauge-theoretic principles, and it takes into account microstructural properties of matter (spin, dilation current, proper hypercharge) as possible physical sources of the gravitational field on an equal footing with macroscopic properties (energy and momentum) of matter. The corresponding spacetime landscape includes as special cases the geometries of Riemann, Riemann-Cartan, Weyl, and Weitzenböck. In the standard formulation of MAG as a gauge theory [14], the gravitational gauge potentials are identified with the metric, coframe, and the linear connection. The corresponding gravitational field strengths are then the nonmetricity, the torsion, and the curvature, respectively. A unified covariant framework for the test body equations of motion in MAG can be found in [16].

As long as no dynamics of the genuine spacetime torsion and nonmetricity is assumed in a particular alternative gravity model, it makes *no* sense to postulate the equation of motion of a test body in the form of an autoparallel. Such a postulate may even be at odds with the actual equations of motion of a theory – which should be derived from its conservation laws, and for the minimal coupling of the structureless matter the resulting trajectory is a Riemannian geodesic curve [7, 16].

It would be a deep delusion to overestimate the two curious examples with artificial torsion and nonmetricity above, and interpret them as an evidence of the physical significance of the autoparallel prescription.

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