Demystifying autoparallels in alternative gravity

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Autoparallel curves along with geodesic curves can arise as trajectories of physical test bodies. We explicitly derive autoparallels as effective post-Riemannian geometric constructs, and at the same time we argue *against* postulating autoparallels as fundamental equations of motion for test bodies in alternative gravity theories.

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I. INTRODUCTION

The equations of motion of test bodies in relativistic gravity are tightly linked to the conservation laws of the theory [1–3]. The explicit derivation of these equations has been intertwined with the development of approximation schemes within relativistic gravity [4–6]. As it is well known, geodesic curves arise as trajectories of structureless test bodies in Riemannian spacetimes with the metric g_{ij} as the gravitational field potential, that determines the metric-compatible Christoffel connection $\widetilde{\Gamma}_{ij}{}^k = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$.

In alternative gravity theories the set of the gravitational field variables is extended to $(g_{ij}, \Gamma_{ij}{}^k)$ where the connection and the metric are no longer compatible, so that the torsion $T_{ij}{}^k := \Gamma_{ij}{}^k - \Gamma_{ji}{}^k$ and the nonmetricity $Q_{kij} := -\nabla_k g_{ij} = -\partial_k g_{ij} + \Gamma_{ki}{}^l g_{lj} + \Gamma_{kj}{}^l g_{il}$ are nontrivial, in general. Autoparallel curves have been *postulated* on several occasions in the literature as candidates for the equations of motion of test bodies in alternative gravity theories. Such ad-hoc postulates, unsubstantiated by the conservation laws, usually lead to inconsistencies with the field equations. Consequently one should abstain from the practice of postulating equations of motion instead of deriving them [7]. With this warning in mind, we here report on two *special* cases, in which autoparallel curves actually do emerge in theories with post-Riemannian spacetime structure.

Let us consider the dynamics of massive particles under the action of the gravitational g_{ij} and a scalar ϕ field. We demonstrate that it is possible to recast the latter into a geometric property of the underlying spacetime, and construct an effective torsion $T_{ij}{}^k$ and nonmetricity Q_{kij} from this scalar field. As a preliminary step, we recall that the deviation of the spacetime geometry from the Riemannian [8] one is described by the distorsion tensor which measures the difference of connections, $N_{ij}{}^k := \widetilde{\Gamma}_{ij}{}^k - \Gamma_{ij}{}^k$. Explicitly:

$$N_{ij}{}^k = -\frac{1}{2}\left(T_{ij}{}^k - T_j{}^k{}_i + T^k{}_{ij}\right) + \frac{1}{2}\left(Q^k{}_{ij} - Q_{ij}{}^k - Q_{ji}{}^k\right). \quad (1)$$

II. EFFECTIVE TORSION FROM THE SCALAR FIELD

At first, we set the nonmetricity equal zero $Q_{kij} = 0$, and introduce the torsion tensor of the form

$$T_{ij}{}^k = \delta_i^k V_j - \delta_j^k V_i, \quad (2)$$

where the vector field

$$V_i = \xi \partial_i \phi \quad (3)$$

is the gradient of the scalar field ϕ with some arbitrary parameter ξ. By construction, the torsion (2) has only one irreducible part, namely, the trace $T_{ki}{}^k = 3V_i$.

Accordingly, we derive from (2) and (1) the distorsion (which is equal to the contortion tensor in this case):

$$N_{ij}{}^k = g_{ij}V^k - \delta_i^k V_j. \quad (4)$$

Now we write down the autoparallel equation for a point particle with the velocity $u^i = \frac{dx^i}{ds}$:

$$\frac{Du^k}{ds} = \frac{d^2x^k}{ds^2} + \Gamma_{ij}{}^k u^i u^j = 0. \quad (5)$$

Substituting $\Gamma_{ij}{}^k = \widetilde{\Gamma}_{ij}{}^k - N_{ij}{}^k$ and using (4), we recast (5) into

$$\frac{d^2x^k}{ds^2} + \widetilde{\Gamma}_{ij}{}^k u^i u^j - \left(V^k - u^k u^i V_i\right) = 0. \quad (6)$$

With the help of (3), we finally rewrite (6) as

$$\frac{d^2x^k}{ds^2} + \widetilde{\Gamma}_{ij}{}^k u^i u^j = \xi\left(g^{ik} - u^k u^i\right)\partial_i\phi. \quad (7)$$

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III. EFFECTIVE NONMETRICITY FROM THE SCALAR FIELD

Another non-Riemannian interpretation of the equation of motion (7) can be achieved by setting the torsion equal zero $T_{ij}{}^k = 0$, and by using the nonmetricity of the form

$$Q_{kij} = -\, g_{ij} V_k + g_{k(i} V_{j)}. \tag{8}$$

As becomes clear from (8), such nonmetricity has two nontrivial irreducible parts: the 3rd and the 4th (Weyl vector). Substituting this into (1), we find the distorsion tensor

$$N_{ij}{}^k = g_{ij} V^k - \delta^k_{(i} V_{j)}. \tag{9}$$

As a result, we discover that the autoparallel equation (5) again reads

$$\frac{d^2 x^k}{ds^2} + \widetilde{\Gamma}_{ij}{}^k u^i u^j - \left(V^k - u^k u^i V_i\right) = 0. \tag{10}$$

Then, by identifying, as before, the vector field with the gradient of the scalar field (3), we finally obtain the same equation of motion (7).

IV. CONCLUSION

In order to demonstrate the physical relevance of the equation of motion above, let us consider the modified action of a structureless point particle

$$I = \int \varphi\, ds = \int \varphi(x) \sqrt{g_{ij}(x) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}\, d\lambda. \tag{11}$$

Such a nonminimal coupling, where the function φ can depend arbitrary on the spacetime coordinates (either directly, or via the geometric invariants), can be viewed as a description of the motion of a body with a variable mass [9].

The variation with respect to the coordinates $x^k(s)$ of the particle is straightforwardly computed

$$\begin{aligned}
\delta I &= \int ds \left[\varphi\left(\frac{1}{2}\delta g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} + g_{ij}\frac{dx^i}{ds}\frac{d\delta x^j}{ds}\right) + \delta\varphi\right] \\
&= \int ds\, \delta x^k \left[\frac{1}{2}\varphi(\partial_k g_{ij}) u^i u^j - \frac{d}{ds}(\varphi g_{ki}) u^i - \varphi g_{ki}\frac{du^i}{ds} + \partial_k \varphi\right] \\
&= -\int ds\, g_{ik}\delta x^k \left[\varphi\left(\frac{du^i}{ds} + \widetilde{\Gamma}_{mn}{}^i u^m u^n\right) - \partial^i\varphi + \frac{d\varphi}{ds}u^i\right].
\end{aligned} \tag{12}$$

Accordingly, the minimal action principle $\delta I = 0$ yields the equation of motion

$$\frac{d^2 x^i}{ds^2} + \widetilde{\Gamma}_{mn}{}^i u^m u^n = \left(g^{ij} - u^i u^j\right) \partial_j \log\varphi. \tag{13}$$

By setting $\varphi = e^{\xi\phi}$, we thus conclude that formally there exist two possibilities to recast the equation of motion (13) into an *autoparallel* in a non-Riemannian spacetime: with the effective torsion (2) or with the effective nonmetricity (8). In principle, one could even think of a combination of the two cases into a more general spacetime geometry with both torsion and nonmetricity.

It is instructive to compare the above results to the equations of motion of a spinless particle nonminimally coupled to the gravitational field, derived from first principles on the basis of the conservation laws [10–12]. Therein, by means of multipolar techniques it was shown, for a very large class of gravitational theories, that test bodies perform a non-geodesic motion with a "pressure"-like force. At the first sight, the resulting equations are equivalent to the autoparallel curve (5) found above. However, one should be clear about the fact that this similarity is *purely formal*. In contrast to the derivation given in [10–12], the effective torsion (2) and the nonmetricity (8) are *not* linked to any kind of gravitational field equations.

Alternative theories of gravity can be studied in the unified framework of the metric-affine gravity (MAG) [13]. The latter is based on gauge-theoretic principles, and it takes into account microstructural properties of matter (spin, dilation current, proper hypercharge) as possible physical sources of the gravitational field on an equal footing with macroscopic properties (energy and momentum) of matter. The corresponding spacetime landscape includes as special cases the geometries of Riemann, Riemann-Cartan, Weyl, and Weitzenböck. In the standard formulation of MAG as a gauge theory [13], the gravitational gauge potentials are identified with the metric, coframe, and the linear connection. The corresponding gravitational field strengths are then the nonmetricity, the torsion, and the curvature, respectively. A unified covariant framework for the test body equations of motion in MAG can be found in [14].

As long as no dynamics of the genuine spacetime torsion and nonmetricity is assumed in a particular alternative gravity model, it makes *no* sense to postulate the equation of motion of a test body in the form of an autoparallel. Such a postulate may even be at odds with the actual equations of motion of a theory – which should be derived from its conservation laws, and for the minimal coupling of the structureless matter the resulting trajectory is a Riemannian geodesic curve [7, 14].

It would be a deep delusion to overestimate the two curious examples with artificial torsion and nonmetricity above, and interpret them as an evidence of the physical significance of the autoparallel prescription.

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