

# The Relativistic Geoid

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**Abstract**—Based on the formalism of General Relativity, we analyze generalizations of concepts used in conventional geodesy. One such concept is the Earth’s geoid. We present our definition of the relativistic geoid in terms of the level sets of a time-independent redshift potential. Such a potential exists for any congruence of Killing observers, i.e. for any rigidly moving object associated with a stationary spacetime in the outer region. The level surfaces of the redshift potential foliate the three dimensional space into isochronometric surfaces, which can be determined with the help of standard clocks. Two such clocks on the same surface will show zero redshift when their frequencies are compared. One of these level surfaces, singled out by a suitable convention, defines the relativistic geoid in our framework.

At the same time, the redshift potential is also an acceleration potential for the congruence of observers. Hence, the isochronometric surfaces are orthogonal to the acceleration of freely falling objects, i.e. they are orthogonal to the local plumb line. Therefore, two independent kinds of measurements can contribute to the determination of the relativistic geoid.

It can be shown that clocks, which are connected by optical fiber links, can be used to determine the redshift potential; this gives the operational foundation of our framework. Moreover, we show that our definition reduces to the well-known Newtonian and post-Newtonian notions in the respective limits. To illustrate our framework, we consider analytic examples of spacetimes, for which we calculate the level surfaces of the redshift potential and illustrate their intrinsic geometry by an embedding into flat Euclidean space.

We emphasize that our definition of the geoid in terms of relativistic concepts is valid for arbitrarily strong gravitational fields. We do not use any approximation in the sense of weak fields or post-Newtonian expansion schemes. Hence, the definition can also be applied to very compact objects such as neutron stars.

**Index Terms**—Geodesy, Earth, Relativistic effects, Astrophysics.

## I. INTRODUCTION

ONE main difference between the Newtonian theory of gravity and Einstein’s General Relativity (GR) is the influence of i) the state of motion and ii) the position in curved spacetime, i.e. in the gravitational field, on the frequencies of clocks. Whereas effects related to i) do also appear in Special Relativity, the effects of category ii) are genuine gravitational effects. In GR, the two separate concepts of space and (absolute) time, as taken for granted in Newtonian gravity, now merge into a curved spacetime. As a consequence, one single scalar potential is not sufficient anymore to describe the gravitational field to full extend and must be replaced by the spacetime metric.

Since the structure of the gravitational theory changes dramatically when passing from Newtonian to relativistic gravity, the fundamental concepts and notions, which are used in

applied sciences such as geodesy, need to be reconsidered and generalized. One of the main concepts in conventional (non-relativistic) geodesy is the geoid. As defined by, e.g., the U.S. National Geodetic Survey, the geoid is “the equipotential surface of the Earth’s gravity field which best fits, in a least squares sense, global mean sea level”. This definition will be addressed briefly in Sec. II by elaborating its underlying assumptions.

As of today, clocks are among the most accurate and advanced measurement devices that modern technology allows to construct. Hence, a relativistic redefinition of concepts such as the Earth’s geoid in terms of notions related to clock measurements might be realizable with high accuracy. In Ref. [1], we have worked out such a definition in terms of a time-independent redshift potential in great detail. This potential a) exists for Killing observers, b) determines the redshift between any two members in a congruence of these observers, c) is also an acceleration potential for the congruence, and d) reproduces the known Newtonian and post-Newtonian expressions in the appropriate limits. As a consequence of a), our definition of the geoid is valid for any central object that is associated with a stationary spacetime. We would like to emphasize that we do not involve approximations such as weak fields for the definition. Hence, this geoid is well-defined also for compact objects such as neutron stars, where we also speak of the geoid for lack of a better word. Properties b) and c) imply that two independent kinds of measurements, which are related to redshifts between clocks and local plumb lines, contribute to the determination of the relativistic geoid.

In Sec. III we give our definition of the relativistic geoid in terms of isochronometric surfaces and we comment on the properties a), b), and c). Section IV addresses property d) by considering the appropriate limits in a top-down approach from GR to (post-)Newtonian gravity. Finally, Sec. V contains examples and we investigate the level surfaces for some spacetimes, of which the metric is analytically known.

For a comparison of different definitions of the relativistic geoid, also in a post-Newtonian setting, see Ref. [1] and references therein. Furthermore, we suggest Refs. [2] and [3] for a review of methods used in relativistic geodesy and its post-Newtonian treatment.

## II. NEWTONIAN GEOID

The Earth’s shape, inner composition, and gravity field show an enormous complexity. As early as 1828 the German mathematician C. F. Gauss introduced the description of a mathematical figure of the Earth to overcome some of the difficulties. To describe this mathematical Earth, one may use the level surfaces of the potential

$$W := U + V, \quad (1)$$

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where  $U$  is the Newtonian gravitational potential, and  $V$  is the centrifugal potential caused by the Earth's rotation. The gravitational part is usually decomposed into spherical harmonics, cf. Refs. [4] and [5],

$$U = -\frac{GM}{r} \sum_{l=0}^{\infty} \sum_{m=0}^l \left(\frac{R_E}{r}\right)^l P_{lm}(\cos\vartheta) [C_{lm} \cos(m\varphi) + S_{lm} \sin(m\varphi)]. \quad (2)$$

In case of axial symmetry the decomposition reduces to

$$U = -G \sum_{l=0}^{\infty} N_l \frac{P_l(\cos\vartheta)}{r^{l+1}}. \quad (3)$$

Here,  $M$  is the mass of the Earth,  $R_E$  is some reference radius (e.g. the equatorial radius of the Earth),  $(r, \vartheta, \varphi)$  are geocentric spherical coordinates,  $P_l$  ( $P_{lm}$ ) are the (associated) Legendre polynomials, and  $C_{lm}, S_{lm}, N_l$  are the multipole coefficients.

In 1873 Listing coined the name geoid for one special equipotential surface that is, by convention, commonly chosen to be closest to mean sea level. In conventional geodesy, one needs to make three assumptions that lead to the definition of a time-independent geoid [1] :

- (A1) The Earth is in rigid motion.
- (A2) The Earth rotates with constant angular velocity about a fixed rotation axis.
- (A3) There are no external forces acting on the Earth.

To cast these assumptions into mathematics, we use the transformation formula from an inertial system  $\Sigma$  to a reference system  $\Sigma'$  attached to a rigidly moving body,

$$\vec{x} = \vec{x}_0(t) + \mathbf{R}(t) \vec{x}'. \quad (4)$$

The position of the center of mass of the central body in the inertial system  $\Sigma$  is described by  $\vec{x}_0(t)$ . The orthogonal matrix  $\mathbf{R}(t)$  describes the momentary rotation of the central body about an axis through its center of mass. Taking successive derivatives of Eq. (4) with respect to  $t$ , keeping  $\vec{x}'$  fixed, yields the velocity and acceleration as measured in the inertial system  $\Sigma$ . Since the matrix  $\mathbf{R}(t)$  is orthogonal, i.e.  $\mathbf{R}(t)^{-1} = \mathbf{R}(t)^T$ , the matrix

$$\boldsymbol{\omega}(t) = \dot{\mathbf{R}}(t) \mathbf{R}(t)^{-1}, \quad (5)$$

which appears in the expressions for velocity and accelerations, is antisymmetric.

As shown in Ref. [1], the three assumptions (A1), (A2), and (A3) imply that:

- (A1') The velocity gradient  $\nabla \otimes \vec{v}$  is antisymmetric.
- (A2')  $\dot{\boldsymbol{\omega}} = 0$ .
- (A3')  $\dot{\vec{a}} = \boldsymbol{\omega} \vec{a}$ .

These conditions have relativistic analogs related to the motion of an isometric congruence in GR [1]. We will outline these analogs in Sec. III.

### III. RELATIVISTIC GEOID

We use SI units and a spacetime metric  $g = g_{\mu\nu} dx^\mu dx^\nu$ . Greek indices run from 0 to 3, and we use Einstein's summation convention. The metric signature is  $(-, +, +, +)$ , the

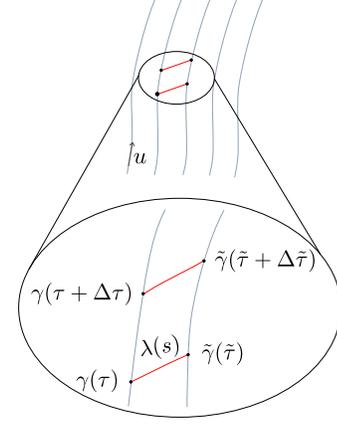


Fig. 1. Sketch for the definition of the redshift. Light signals are exchanged between two worldlines  $\gamma$  and  $\tilde{\gamma}$ , which are members of a congruence with tangent vector field  $u$ .

speed of light is denoted by  $c$ , and  $G$  is Newton's gravitational constant.

In 1985, Bjerhammar [6] was the first to give a definition of the relativistic geoid:

*The relativistic geoid is the surface nearest to mean sea level on which precise clocks run with the same speed.*

In the following, we translate Bjerhammars wording, which is not precise (enough) in our view, into the formalism of GR. We suggest that “precise clocks” are taken to be standard clocks. These are defined by the normalization of the observers tangent vector. Equivalently, standard clocks can be characterized operationally as described in Ref. [7]. Now, we still need to define the conditions for clocks to “run with the same speed”. Instead of being related to some synchronization procedure, which is tedious and not even possible in general, we use the redshift between clocks.

Given a congruence of observer worldlines with normalized tangent vector field  $u$ , we pick any two members  $\gamma$  and  $\tilde{\gamma}$  of the congruence. Since the observers use standard clocks, we have  $g(u, u) = -c^2$ , and the two chosen worldlines are parametrized by the respective proper times  $\tau$  and  $\tilde{\tau}$ . In GR, the redshift  $z$  between the two worldlines is given by [8]

$$z + 1 = \frac{\nu}{\tilde{\nu}} = \frac{d\tilde{\tau}}{d\tau} = \lim_{\Delta\tau \rightarrow 0} \frac{\Delta\tilde{\tau}}{\Delta\tau} = \frac{\left( g_{\mu\nu} \frac{d\lambda^\mu}{ds} \frac{d\gamma^\nu}{d\tau} \right) \Big|_{\gamma(\tau)}}{\left( g_{\rho\sigma} \frac{d\lambda^\rho}{ds} \frac{d\tilde{\gamma}^\sigma}{d\tilde{\tau}} \right) \Big|_{\tilde{\gamma}(\tilde{\tau})}}. \quad (6)$$

Here,  $\lambda$  is a lightlike geodesic, with an affine parametrization  $s$ , connecting the events on  $\gamma$  and  $\tilde{\gamma}$  as sketched in Fig. 1. The redshift is related to the frequency ratio  $\nu/\tilde{\nu}$  and these frequencies can be understood as the rates of emission and reception of signals, respectively.

If a scalar field  $\phi$  exists with the property that its difference between any two observer worldlines of the congruence yields the redshift, i.e. with the property that

$$\log(z + 1) =: \phi(\tilde{\gamma}(\tilde{\tau})) - \phi(\gamma(\tau)), \quad (7)$$

we call  $\phi$  a redshift potential. Since we want to establish a generalization of the time-independent geoid, we seek for a time-independent redshift potential. In [9], it was shown that  $\phi$  indeed has this property if and only if

$$e^\phi u := \xi \quad (8)$$

is a Killing vector field. Since the congruence tangent vector field  $u$  is timelike, the Killing vector field must be timelike as well. Hence, the spacetime must be stationary. Thereupon, we can introduce coordinates  $(t, x^i)$  with  $\xi = \partial_t$  such that the metric reads

$$g = e^{2\phi(x)} \left[ -(c dt + \alpha_a(x) dx^a)^2 + \alpha_{ab}(x) dx^a dx^b \right]. \quad (9)$$

All metric functions  $\phi$ ,  $\alpha_a$ , and  $\alpha_{ab}$  depend on the spatial position but not on the time  $t$ . For a stationary spacetime with a metric in the form above, the redshift potential  $\phi$  is given by the relation

$$c^2 e^{2\phi} = -g_{\mu\nu} \xi^\mu \xi^\nu = -g_{tt}. \quad (10)$$

Therefore, the redshift between any two integral curves of  $u = \exp(-\phi)\xi$  is

$$z + 1 = \frac{\nu}{\bar{\nu}} = e^{\phi|_{\bar{\gamma}} - \phi|_{\gamma}} = \frac{e^{\phi|_{\bar{\gamma}}}}{e^{\phi|_{\gamma}}} = \frac{\sqrt{-g_{tt}|_{\bar{\gamma}}}}{\sqrt{-g_{tt}|_{\gamma}}}. \quad (11)$$

The level sets of  $\phi$  foliate the three dimensional space into isochronometric surfaces. Two members of the observer congruence on the same surface measure zero redshift between their clocks. For observers on different level sets the redshift is given by the potential difference, see Eq. (11).

We now suggest the following definition of the relativistic geoid [1]:

*The relativistic geoid is the level surface of the redshift potential  $\phi$  that is closest to mean sea level.*

These level surfaces can be determined with the help of standard clocks by pairwise frequency comparison. In Ref. [1], it was shown that these clocks may also be connected by optical fiber links instead of only exchanging signals along lightlike geodesics.

As we will outline in the next section, the Newtonian limit of our definition reduces to the conventional (non-relativistic) definition given in Sec. II. Moreover, in the post-Newtonian approximation of GR, the definition of Soffel *et al.* [10] is recovered.

We end this section by stating the relativistic analogs of the Newtonian requirements (A1'), (A2'), and (A3') for the definition of a time-independent geoid. A time-independent redshift potential exists for any congruence of Killing observers, and these congruences have important properties. The motion of a congruence of timelike worldlines with normalized tangent vector field  $u$  can be decomposed kinematically into rotation, shear, and expansion, see e.g. Refs. [11] and [12]. Since we can think of the congruence as being realized by the worldlines of observers on the Earth's surface, a natural generalization of the condition (A1') is to consider a Born-rigid congruence, i.e. a congruence with vanishing shear and expansion. For such a congruence, all relative distances between the observers remain constant over time, as it should be for observers on the

surface of a rigidly moving object. This first condition now leads to [1]

$$(A1'') P_\mu^\rho P_\nu^\sigma D_{[\sigma} u_{\rho]} = 0,$$

which states that the symmetric part of the covariant derivative in the rest space of the congruence vanishes. Hence, only the anti-symmetric part remains. Here,  $P$  is the projection operator defined by  $c^2 P_\nu^\mu = c^2 \delta_\nu^\mu + u^\mu u_\nu$ . The remaining anti-symmetric part of the covariant derivative is called the rotation,

$$\omega_{\mu\nu} := P_\mu^\rho P_\nu^\sigma D_{[\sigma} u_{\rho]}. \quad (12)$$

As the dual to  $\omega_{\mu\nu}$ , in the rest space of  $u$ , we can introduce the purely spatial rotation vector

$$\omega^\mu := \frac{1}{2c} \eta^{\mu\nu\sigma\lambda} u_\nu \omega_{\sigma\lambda} = \frac{1}{c} \eta^{\mu\nu\sigma\lambda} u_\nu \partial_\lambda u_\sigma. \quad (13)$$

Its magnitude gives the angular velocity. The Newtonian requirements (A2') and (A3') now translate into the following conditions [1]:

$$(A2'') P_\nu^\mu \dot{\omega}^\nu = 0.$$

$$(A3'') P_\nu^\mu \dot{a}^\nu = \omega^\mu{}_\nu a^\nu.$$

Because of condition (A2''), the rotation vector  $\omega^\mu$  is Fermi-Walker transported and its magnitude is constant along each worldline. This states that the rotation axis and the angular velocity are time independent. As can be seen by condition (A3''), the change of the acceleration is only due to the rotation. The acceleration vector will always points to the same neighboring congruence member.

In Ref. [11], Ehlers has shown that, for a congruence with properties (A1''), (A2'') and (A3''), these three conditions together are equivalent to  $D_{[\nu} a_{\mu]} = 0$ . Hence, there must exist a scalar quantity of which the gradient determines the acceleration,

$$a_\mu = c^2 \partial_\mu \phi. \quad (14)$$

For a Born-rigid congruence this is equivalent to the fact that  $u$  is given by  $e^\phi u = \xi$ , where  $\xi$  is a Killing vector field [13]. Hence, the redshift potential and acceleration potential  $\phi$  coincide.

#### IV. NEWTONIAN AND POST-NEWTONIAN LIMITS

In the following, we demonstrate that our definition of the relativistic geoid reduces to the definition given by Soffel *et al.* in 1988 [10] in the post-Newtonian limit of GR.

In co-rotating geocentric coordinates  $(c\bar{T}, \bar{X}^i)$  the metric used in the work by Soffel *et al.* [10] is

$$g_{00} = - \left( 1 - \frac{2U}{c^2} + \frac{2U^2}{c^4} \right) + \Omega^2 (\bar{X}^2 + \bar{Y}^2) / c^2, \quad (15a)$$

$$g_{0i} = \mathbf{L} - \bar{\mathbf{X}} \times \boldsymbol{\Omega} / c, \quad (15b)$$

$$g_{ij} = \delta_{ij} \left( 1 + \frac{2U}{c^2} \right), \quad (15c)$$

where

$$\mathbf{L} = -2G \frac{\mathbf{J} \times \bar{\mathbf{X}}}{c^3 R^3}. \quad (16)$$

The angular velocity and angular momentum of the Earth are given by  $\Omega$  and  $\mathbf{J}$  and the gravitational potential  $U$  fulfills the equation  $\Delta U(X) = -4\pi G\rho(X)$ . The (mass) density  $\rho$  is related to the energy-momentum tensor of the Earth by  $\rho = (T^{00} + T^{ii})/c^2$ . Here, we use the three-vector notation only as a shorthand notation. As can be read of from the metric (15), the vector field  $\partial_{\bar{T}}$  is a Killing vector field. Observers on the Earth's surface move on its integral curves because we use co-rotating coordinates; for them  $d\bar{X}^i = 0$ . These observers form an isometric congruence. The corresponding relativistic redshift (and acceleration) potential  $\phi_{PN}$  is given by

$$e^{2\phi_{PN}} = -g_{00} = 1 - \frac{2U}{c^2} + \frac{2U^2}{c^4} - \Omega^2(\bar{X}^2 + \bar{Y}^2)/c^2. \quad (17)$$

The relativistic geoid in our framework is defined by one level surface of this potential, i.e. it is defined by the condition

$$U + \frac{1}{2}\Omega^2(\bar{X}^2 + \bar{Y}^2) - \frac{U^2}{c^2} = \text{constant}. \quad (18)$$

This is Eq. (4) in the article by Soffel *et al.*, see Ref. [10]; it defines the post-Newtonian geoid. Hence, our definition reduces to this definition in the post-Newtonian approximation of GR.

The first two terms in Eq. (18) reproduce the classical definition of the Newtonian geoid, stating that the sum of the gravitational and centrifugal potential should be constant, cf. Eq. (1). Hence, in the pure Newtonian limit, our definition also reproduces the one used in conventional geodesy. The last term in Eq. (18) describes the relativistic correction at the first order post-Newtonian level.

The fact that the redshift potential  $\phi$  is also an acceleration potential for the isometric congruence generalizes the equality of the a- and u-geoid, as defined by Soffel *et al.* in Ref. [10], to full General Relativity.

## V. EXAMPLES

To illustrate our framework and definition of the relativistic geoid, we investigate the isochronometric surfaces in different spacetimes. Here, we summarize the results for a) the Schwarzschild spacetime, b) the Kerr spacetime, c) the Erez-Rosen spacetime, and d) the q-metric spacetime; for details see Ref. [1]. All these spacetimes have a mass monopole moment, related to the total mass of the central object. The last three examples do also possess an independent mass quadrupole moment, which leads to nontrivial (i.e. non spherical) geometry. Note also that for the Kerr spacetime there is a spin dipole moment as well, which is coupled to the mass quadrupole. For all relativistic multipole moments, we use the definition given by Geroch [14] and Hansen [15].

In the following, we give the expressions for the isochronometric surfaces in the spacetimes mentioned above. To visualize the geometry of these surfaces one may use a certain contour plot. However, this is strongly biased by the respective coordinate system, which is chosen to put the spacetime metric into a desired form. To overcome this arbitrariness, we isometrically embed the level surfaces of the redshift potential into flat Euclidean space  $\mathbb{R}^3$ . If the embedding exists, it is

unique. The geometry of a chosen surface in the embedding space, i.e. distances and angles, is the same as the geometry of the respective surface in the spacetime manifold. For details of the embedding we refer to the appendix of Ref. [1]. The results are shown in Fig. 2 for all examples and oblate configurations (negative quadrupole moment). We show the level sets for rotating observers, which can be imagined as being fixed to the surface of a rigidly rotating central object such as the Earth.

Although all our examples are axisymmetric situations, our definition is of course not restricted to axisymmetry. However, we believe that the following examples are instructive and give insight into the presented framework.

### A. Schwarzschild spacetime

The Schwarzschild spacetime is, due to Birkhoff's theorem, the unique solution for the spacetime outside a spherically symmetric mass distribution. The metric reads

$$g = -\left(1 - \frac{2m}{r}\right) c^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2, \quad (19)$$

where  $2m = 2GM/c^2 =: r_s$  is the Schwarzschild radius. The spacetime has only a non-vanishing mass monopole  $M_0 = M$ .

From the form (19) of the metric we read that  $\partial_t$  and  $\partial_\varphi$  are Killing vector fields. Whilst the first one is timelike, the latter is spacelike. Hence, we can construct two different congruences of Killing observers: I) observers of which the worldlines are integral curves of  $\partial_t$ , referred to as static (stationary in the case of Kerr spacetime) observers, and II) observers of which the worldlines are integral curves of  $\partial_t + \Omega \partial_\varphi$ , referred to as rotating observers. In the following, all quantities associated with the first congruence are indicated by the subscript  $(\cdot)_{\text{stat}}$ , whereas quantities associated with the second congruence are indicated by the subscript  $(\cdot)_{\text{rot}}$ . Note that, in order to keep the combination timelike, the values of  $\Omega \in \mathbb{R}$  for congruence II) are restricted.

The isochronometric surfaces, as seen by the respective observers, are given by level sets of the expressions [1]

$$e^{2\phi_{\text{stat}}} = \left(1 - \frac{2m}{r}\right), \quad (20a)$$

$$e^{2\phi_{\text{rot}}} = \left(1 - \frac{2m}{r}\right) - \frac{\Omega^2}{c^2} r^2 \sin^2 \vartheta. \quad (20b)$$

### B. Kerr spacetime

The Kerr metric in Boyer-Lindquist coordinates  $(t, r, \vartheta, \varphi)$  reads

$$g = -\left(1 - \frac{2mr}{\rho^2}\right) c^2 dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\vartheta^2 + \sin^2 \vartheta \left(r^2 + a^2 + \frac{2mra^2 \sin^2 \vartheta}{\rho^2}\right) d\varphi^2 - \frac{4mra \sin^2 \vartheta}{\rho^2} c dt d\varphi. \quad (21)$$

The parameter  $m$  is related to the central mass as before. The Kerr parameter  $a$  is given by the angular momentum  $J$  of

the source,  $a = J/(Mc)$ . The spacetime possesses a mass monopole  $M_0 = M$ , a mass quadrupole  $M_2 = -Ma^2$ , and a spin dipole  $J_1 = Ma$ . Since  $\partial_t$  and  $\partial_\varphi$  are Killing vector fields, we can, again, construct the two congruences of Killing observers I) and II). Hence, the redshift potentials, as seen by these observers, are [1]

$$e^{2\phi_{\text{stat}}} = 1 - \frac{2mr}{\rho^2} = 1 - \frac{2mr}{r^2 + a^2 \cos^2 \vartheta}, \quad (22a)$$

$$e^{2\phi_{\text{rot}}} = 1 - \frac{2mr}{r^2 + a^2 \cos^2 \vartheta} + 4 \frac{\Omega}{c} \frac{amr \sin^2 \vartheta}{(r^2 + a^2 \cos^2 \vartheta)} \quad (22b)$$

$$- \frac{\Omega^2}{c^2} \sin^2 \vartheta \left( r^2 + a^2 + \frac{2mra^2 \sin^2 \vartheta}{r^2 + a^2 \cos^2 \vartheta} \right), \quad (22c)$$

respectively. Note that the Kerr parameter  $a$  can be chosen to either fit the angular momentum of the central object or to fit the mass quadrupole. The relativistic geoid is, by definition, one of level surfaces of the potentials above.

### C. Erez-Rosen spacetime

The metric found by Erez and Rosen [16] describes a spacetime with mass monopole  $M_0 = M$  and mass quadrupole  $M_2 = 2/15 (G/c^2)^2 q_2 M^3$ . It is a generalization of the Schwarzschild spacetime, which is recovered in the limit  $q_2 \rightarrow 0$ . Due to axisymmetry, we have the same Killing vector field as before and construct the two types of observer congruences.

In Schwarzschild-like coordinates, the isochronometric surfaces follow from the level surfaces of [1]

$$e^{2\phi_{\text{stat}}} = \left( 1 - \frac{2m}{r} \right) \exp \left\{ q_2 (3 \cos^2 \vartheta - 1) \times \left[ \left( \frac{3}{4} \left( \frac{r}{m} - 1 \right)^2 - \frac{1}{4} \right) \log \left( 1 - \frac{2m}{r} \right) + \frac{3}{2} \left( \frac{r}{m} - 1 \right) \right] \right\}, \quad (23a)$$

$$e^{2\phi_{\text{rot}}} = e^{2\phi_{\text{stat}}} - \frac{\Omega^2}{c^2} r^2 \sin^2 \vartheta e^{-2\phi_{\text{stat}}}. \quad (23b)$$

### D. q-metric spacetime

The q metric spacetime [17], see also [18], is a simple generalization of the Schwarzschild spacetime. It is obtained by a Zipoy-Voorhees transformation of the Schwarzschild solution [19], [20]. Since axisymmetry is preserved, also this spacetime allows to construct the two observer congruences.

The redshift potential, as seen by the static or rotating observers, is, respectively, [1]

$$e^{2\phi_{\text{stat}}} = \left( 1 - \frac{2m}{r} \right)^{1+q}, \quad (24a)$$

$$e^{2\phi_{\text{rot}}} = \left( 1 - \frac{2m}{r} \right)^{1+q} - \frac{\Omega^2}{c^2} \left( 1 - \frac{2m}{r} \right)^{-q} r^2 \sin^2 \vartheta. \quad (24b)$$

## VI. CONCLUSION

We have given a definition of the relativistic geoid in terms of the level sets of a time-independent redshift potential. This relativistic potential exists for any congruence of Killing observers, and is, at the same time, also a potential for the acceleration of this isometric congruence. Hence, the a- and u- geoids, which were defined in a post-Newtonian setting before, coincide also in the full theory of GR. We refer the reader to Ref. [1], where more properties are worked out in great detail.

Our definition of the geoid is valid in full General Relativity; it applies not only to the Earth but to all compact objects associated with a stationary spacetime outside. The definition is based on the assumption that the Earth rotates rigidly with constant angular velocity about a fixed axis. Under this assumption, the Earth is associated with an isometric congruence of worldlines. As the isochronometric surfaces may be realized with networks of standard clocks that are connected by fiber links, this is an operational definition of the geoid. Since the redshift potential is also an acceleration potential, the level surfaces are orthogonal to the local plumb line. This yields a second, independent, measurement prescription to determine the relativistic geoid.

The real gravitational field of the Earth will in practice contain time-dependent effects due to geodynamical phenomena and perturbations from other celestial bodies. Our stationary geoid is to be considered as the time average of the real geoid of the Earth. The time-dependent parts have to be treated through, e.g. an appropriate reduction.

Our formalism is valid for stationary non-axisymmetric objects as well, as long as gravitational radiation and the loss of energy due to the emission can be ignored. For planets such as the Earth, this assumption is certainly justified. As we have shown in Sec. III, our definition reduces to the Newtonian and post-Newtonian definitions in the appropriate limits.

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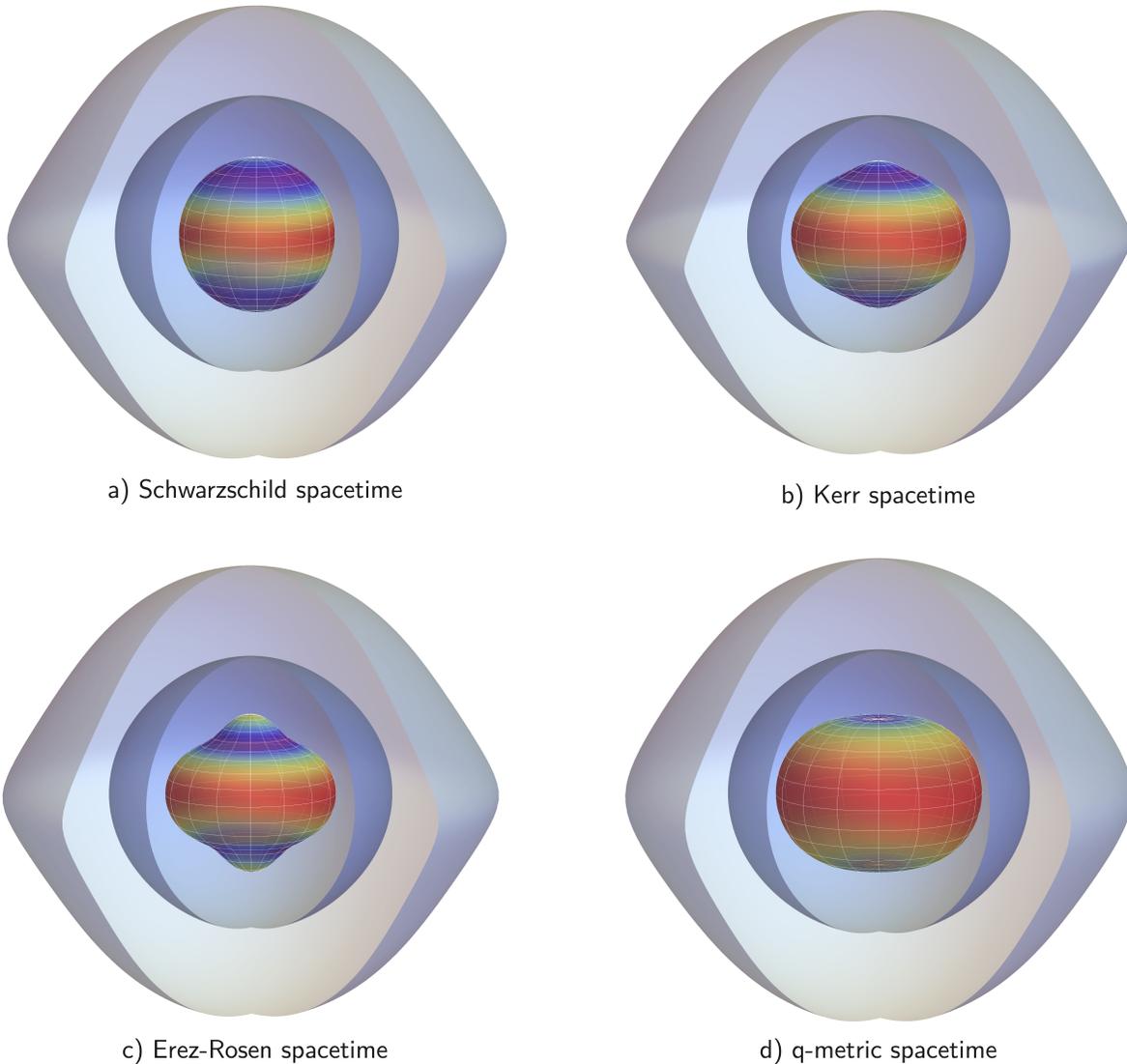


Fig. 2. The isometric embedding of the isochronometric surfaces into flat Euclidean space. We show the geometry of the level sets as seen by rotating observers, of which the worldlines are integral curves of the Killing vector field  $\partial_t + \Omega \partial_\varphi$ , in the respective spacetime. We have chosen the value of  $\Omega$  such that the combination is timelike. The innermost level surfaces is closest to the gravitating object and color-coded to depict its geometry. Here, red corresponds to the the farthest distance and purple to the closest distance to the origin of  $\mathbb{R}^3$ . For the Schwarzschild spacetime, the innermost surface is close to the horizon, whereas for the Kerr spacetime it is close to the outer ergosurface.

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