INTRODUCTION TO GENERAL RELATIVITY AND COSMOLOGY

Living script

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Foreword

This material was prepared by D. Puetzfeld as part of Iowa States Astro 405/505 (Fall 2004) course. Comments and suggestions are welcome! Please report any errors, typos, etc. (probably many) back to me (dpuetz@iastate.edu). You can find the material covered in the single lectures online under www.thp.uni-koeln.de/∼dp (click on teaching). There will also be some small computer algebra programs available on this site. The script only supplements the lecture, it is not a substitute for attending the lecture!

Warning: This is a "living" script, i.e. I will update it occasionally without notice.

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Chapter 1

Basics: General Relativity and Cosmology

1.1 Part I - Fun with tensors

Goal: Provide appropriate formalism for a relativistic formulation of a gravity theory. Physical laws should remain form invariant in different coordinate systems.

1.1.1 Scalars, vectors & tensors

A *scalar* is any physical quantity determined by a single numerical value which is independent of the coordinate system. Examples: (i) Charge and mass of a particle in Newtonian mechanics. (ii) Charge and rest-mass of a particle in special relativity. Simplest example of a vector is a displacement between two points, say A and B. In non-Cartesian coordinates we can reach B from A, with coordinates[1] x^μ, via the infinitesimal displacement $x^\mu + dx^\mu$. The components of such a vector between the points A and B are the differentials dx^μ. This is valid in general coordinates. Starting from the infinitesimal vector AB with components dx^μ we can construct the finite vector v^μ. Consider a curve through A and B determined by n functions of the scalar parameter λ: $x^\mu = f^\mu(\lambda)$. If A and B correspond to the values λ and $\lambda + d\lambda$ the tangent vector v^μ to the curve at A has the components $v^\mu = \frac{dx^\mu}{d\lambda}$. The infinitesimal displacement dx^μ or equivalently v^μ are prototypes for what is called a *contravariant vector*. Lets work out the components of the vector AB under coordinate transformations $x^\mu \to \tilde{x}^\mu$, which are given by n equations of the form

$$\tilde{x}^\mu = f^\mu(x^\nu), \quad \mu, \nu = 1, \ldots, n. \tag{1.1}$$

From these equations we derive

$$d\tilde{x}^\mu = \sum_\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu, \tag{1.2}$$

[1]In this section all greek indices run from from $1, \ldots, n$. Later on we will confine ourselves to 4-dimensional spaces.

Now lets derive the transformation law for the vector v^μ. Since λ is a scalar parameter $d\lambda$ has the same value in both coordinate systems and consequently we find

$$\tilde{v}^\mu = \sum_\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} v^\nu. \tag{1.3}$$

This leads us to the general definition of a contravariant vector.

Contravariant vector A *contravariant vector* is a quantity a^μ with n components depending on the coordinate system in such a way that the components a^μ in the coordinate system x^μ are related to the components $\tilde{a}^\mu$ in the coordinate system $\tilde{x}^\mu$ by a relation of the form

$$\tilde{a}^\mu = \sum_v \frac{\partial \tilde{x}^\mu}{\partial x^\nu} a^\nu. \tag{1.4}$$

Now lets come to another important point. In order to have coordinates x^μ and $\tilde{x}^\mu$, which are equally acceptable, we must be able to solve equation (1.2) with respect to dx^ν. Of course this is only possible if det $\frac{\partial \tilde{x}^\mu}{\partial x^\nu} \neq 0$ or ∞. Since

$$\sum_\alpha \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \tilde{x}^\nu} = \sum_\alpha \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\alpha}{\partial x^\nu} = \delta^\mu_\nu = \begin{cases} 1 & \text{if} \quad \mu = \nu \\ 0 & \text{if} \quad \mu \neq \nu \end{cases} \tag{1.5}$$

we find at once a solution of (1.2) for dx^ν, namely

$$dx^v = \sum_\mu \frac{\partial x^v}{\partial \tilde{x}^\mu} d\tilde{x}^\mu. \tag{1.6}$$

The quantity δ^μ_ν in equation (1.5) is called *Kronecker symbol*.

Covariant vector The quantity b_μ with n components is called a *covariant vector*, if for any contravariant vector a^μ

$$\sum_\mu b_\mu a^\mu = \sum_\mu \tilde{b}_\mu \tilde{a}^\mu \text{ for any } x^\mu \rightarrow \tilde{x}^\mu.$$

This sum is called the scalar product of the vectors a^μ and b_μ. From this definition follows that the transformation rule for b_μ is given by

$$b_\mu = \sum_\nu \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \tilde{b}_\nu, \tag{1.7}$$

additionally, remember (1.5), we have

$$\tilde{b}_\mu = \sum_\nu \frac{\partial x^\nu}{\partial \tilde{x}^\mu} b_\nu. \tag{1.8}$$

Since we now know the difference between upper and lower indices we are going to use the so-called summation convention, i.e. we shall sum automatically over upper and lower indices

$$X^{\dots}_{\dots\alpha} Y^{\dots\alpha}_{\dots} \equiv \sum_{\alpha} X^{\dots}_{\dots\alpha} Y^{\dots\alpha}_{\dots}. \tag{1.9}$$

With this knowledge the definition of tensors of higher order is straightforward.

Contravariant/Covariant/Mixed tensors A *contravariant tensor* $T^{\mu_1\dots\mu_k}$ of rank k is an object which transforms under coordinate transformations in such a way that, for arbitrary covariant vectors $c^{(i)}_\mu$ $(i = 1, \dots, k)$ the sum $T^{\mu_1\dots\mu_k} c^1_{\mu_1} \dots c^k_{\mu_k}$ is a scalar, i.e.

$$T^{\mu_1\dots\mu_k} c^1_{\mu_1} \cdots c^k_{\mu_k} = \tilde{T}^{\mu_1\dots\mu_k} \tilde{c}^1_{\mu_1} \cdots \tilde{c}^k_{\mu_k} \text{ for any } x^\mu \to \tilde{x}^\mu. \tag{1.10}$$

Of course the definitions for a covariant and mixed tensor are straightforward. In the *covariant* case we have

$$T_{\mu_1\dots\mu_k} c^{\mu_1}_1 \cdots c^{\mu_k}_k = \tilde{T}_{\mu_1\dots\mu_k} \tilde{c}^{\mu_1}_1 \cdots \tilde{c}^{\mu_k}_k \quad \text{for any } x^\mu \to \tilde{x}^\mu, \tag{1.11}$$

and in the $k = m + l$ *mixed* case

$$T^{\mu_1\dots\mu_m}{}_{\mu_{m+1}\dots\mu_{l+m}} c^1_{\mu_1} \cdots c^m_{\mu_m} c^{\mu_{m+1}}_1 \cdots c^{\mu_{l+m}}_l = \tilde{T}^{\mu_1\dots\mu_m}{}_{\mu_{m+1}\dots\mu_{l+m}} \tilde{c}^1_{\mu_1} \cdots \tilde{c}^m_{\mu_m} \tilde{c}^{\mu_{m+1}}_1 \cdots \tilde{c}^{\mu_{l+m}}_l \tag{1.12}$$

for any $x^\mu \to \tilde{x}^\mu$. Hence a tensor of rank $(0, 0)$ is a scalar, $(1, 0)$ a contravariant vector, and $(0, 1)$ a covariant vector. Example: Transformation properties of a tensor[2] of rank

[2]Physical example for a tensor of rank 2 is the totally antisymmetric tensor $F^{\alpha\beta}$ of the electromagnetic field:

$$F^{\alpha\beta} := \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

The current four vector $j^\alpha := (\rho,\ j^a)$. Exercise: Show that (i) $F^{(\alpha\beta)} = 0$. (ii) Maxwell's equations take the form:

$$\begin{aligned} \partial_\beta F^{\alpha\beta} &= j^\alpha, \\ \partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} &= 0. \end{aligned}$$

(iii) The latter of the two equations can be written as $\partial_{[\alpha} F_{\beta\gamma]} = 0$. (iv) The continuity equation takes the form $\partial_\alpha j^\alpha = 0$. (v) If we introduce the four potential $\phi^\alpha = (\phi, A^a)$ the tensor of the electromagnetic field can be written as

$$F_{\alpha\beta} = 2\partial_{[\beta}\phi_{\alpha]}.$$

(Remember: $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$). (vi) A gauge transformation of the potential $\phi_\alpha \to \tilde{\phi}_\alpha = \phi_\alpha + \partial_\alpha \psi$ with a scalar ψ does not change $F^{\alpha\beta}$.

2 (for the 3 different cases $(2,0), (0,2), (1,1)$)

$$\tilde{T}^{\alpha\beta} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu} T^{\mu\nu}, \quad \tilde{T}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\beta} T_{\mu\nu}, \quad \tilde{T}^\alpha{}_\beta = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial \tilde{x}^\beta} T^\mu{}_\nu.$$

According to the definitions given so-far we can perform simple algebraic operations with tensors. Tensors of the same order (k,l) can be added, their *sum* being again a tensor of the same order[3]. The *product* of two tensors of the order (k,l) and $\left(\hat{k},\hat{l}\right)$ will be a tensor of order $\left(k+\hat{k}, l+\hat{l}\right)$[4]. A *contraction* is possible for any tensor of the order (k,l) with $k,l > 0$. Putting $\mu_i = \nu_j$ in a tensor $T^{\mu_1 \dots \check{\mu}_i \dots \mu_k}{}_{\nu_1 \dots \check{\nu}_k \dots \nu_l} = T^{\mu_1 \dots \alpha \dots \mu_k}{}_{\nu_1 \dots \alpha \dots \nu_l}$ yields a tensor of rank $(k-1, l-1)$. Example: Contraction of a tensor of the order $(2,1)$, i.e. $T^{\alpha\beta}{}_\mu$, yields a tensor of rank $(1,0) : T^\alpha := T^{\alpha\beta}{}_\beta$[5]. The scalar $T^\alpha{}_\alpha$ is called the *trace* of a mixed tensor $T^\alpha{}_\beta$. Sometimes it is useful to split up tensors in the symmetric and antisymmetric part. The *symmetric* and *antisymmetric* part of a tensor of rank $(0,2)$ is defined by

$$T_{(\alpha\beta)} := \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right), \quad T_{[\alpha\beta]} := \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right).$$

The (anti)symmetry property of a tensor will be conserved in all frames[6]. We call a tensor of rank $(0,2)$ totally symmetric (antisymmetric) if $T_{\alpha\beta} = T_{(\alpha\beta)}$ $\left(T_{\alpha\beta} = T_{[\alpha\beta]}\right)$. The same statements are of course true for a tensor of rank $(2,0)$. Warning: No symmetry properties can be defined for a mixed tensor $T^\alpha{}_\beta$.The matrix of the components of $T^\alpha{}_\beta$ would be symmetric in some frame x^μ, but this property would not be conserved in a coordinate transformation[7]. Of course it is possible to generalize the concept of (anti)symmetry to tensors with higher rank. Example: We will call a tensor of rank $(0,3)$ (anti)symmetric if

$$\begin{aligned} T^{\alpha\beta\mu} &= T^{\beta\alpha\mu} = T^{\alpha\mu\beta} = T^{\mu\beta\alpha}, \\ T^{\alpha\beta\mu} &= -T^{\beta\alpha\mu} = -T^{\alpha\mu\beta} = -T^{\mu\beta\alpha}. \end{aligned}$$

Its totally (anti)symmetric part is given by

$$\begin{aligned} T_{[\alpha\beta\mu]} &= \frac{1}{6}\left(T_{\alpha\beta\mu} + T_{\mu\beta\alpha} + T_{\mu\alpha\beta} + T_{\mu\beta\alpha} + T_{\beta\alpha\mu} + T_{\alpha\mu\beta}\right), \\ T_{(\alpha\beta\mu)} &= \frac{1}{6}\left(T_{\alpha\beta\mu} + T_{\mu\beta\alpha} + T_{\mu\alpha\beta} - T_{\mu\beta\alpha} - T_{\beta\alpha\mu} - T_{\alpha\mu\beta}\right). \end{aligned}$$

[3]Exercise: Show this explicitly for two $(1,0)$ tensors, i.e. the sum of two contravariant vectors.
[4]Exercise: Show this explicitly for two $(1,0)$ tensors, i.e. the product of two contravariant vectors.
[5]Exercise: Verify that T^α, i.e. $T^{\alpha\beta}{}_\beta$, really transforms like a contravariant vector.
[6]Exercise: Show this explicitly.
[7]Exercise: Show that the number of independent components of the symmetric tensor $T_{(\alpha\beta)}$ equals $\frac{n(n+1)}{2}$ and of the antisymmetric tensor $T_{[\alpha\beta]}$ equals $\frac{n(n-1)}{2}$.

1.1.2 Covariant derivative & connection

Consider a region of space M on which some tensor $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l}$ is given at each point $P(x^\nu)$ of M, i.e. $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l}(x^\nu)$. In this case we call $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l}$ a *tensor field* on M. Now we can ask the question whether it is possible to construct new tensor fields by differentiating the given one. If we start with the simplest tensor field $\phi = \phi(x^\nu)$ of rank $(0,0)$ the derivatives of this field $\phi_{,\mu} := \frac{\partial \phi}{\partial x^\mu}$ are the components of a covariant vector as immediately becomes clear from

$$d\phi = \phi_{,\mu}\, dx^\mu.$$

(Since $d\phi$ is a scalar and dx^μ an arbitrary contravariant vector). Next consider a tensor field of rank $(0,1)$, i.e. a covariant vector a_μ. The derivative $a_{\mu,\nu} := \frac{\partial a_\mu}{\partial x^\nu}$ has the following form in some other coordinate system

$$\tilde{a}_{\mu,\nu} \equiv \frac{\partial \tilde{a}_\mu}{\partial \tilde{x}^\nu} = \frac{\partial}{\partial \tilde{x}^\nu}\left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu} a_\alpha\right) = \frac{\partial^2 x^\alpha}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu} a_\alpha + \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} a_{\alpha,\beta}. \tag{1.13}$$

This is almost the coordinate transformation formula for a covariant tensor of rank $(0,2)$. The first term would vanish in case of a linear transformation, but since we are interested in arbitrary coordinates $a_{\mu,\nu}$ is not a tensor. In order to overcome the problem with the transformation rule in equation (1.13) we rewrite it with the help of (1.7), we have

$$\tilde{a}_{\mu,\nu} - \frac{\partial^2 x^\alpha}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\beta}{\partial x^\alpha}\tilde{a}_\beta = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} a_{\alpha,\beta}. \tag{1.14}$$

Now let us assume that there is a tensor $T_{\mu\nu}$ whose components in the coordinate system x^μ are the derivatives $a_{\mu,\nu}$. Hence we can reinterpret the lhs of (1.14) as the components of $T_{\mu\nu}$ in the coordinate system $\tilde{x}^\mu$, i.e.%

$$\tilde{T}_{\mu\nu} = \tilde{a}_{\mu,\nu} - \frac{\partial^2 x^\alpha}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\beta}{\partial x^\alpha}\tilde{a}_\beta := \tilde{a}_{\mu,\nu} - \tilde{\gamma}^\beta_{\mu\nu}\tilde{a}_\beta.$$

Therefore there might be coordinate systems in which the derivatives $a_{\alpha,\beta}$ are the components of the tensor, but we have to introduce a new coordinate dependent quantity $\gamma^\beta_{\mu\nu}$ to cover also the general case. If one now assumes that $T_{\mu\nu}$ is a tensor in all coordinate systems of the form

$$T_{\mu\nu} = a_{\mu,\nu} - \Gamma^\alpha_{\ \mu\nu} a_\alpha \text{ in } x^\mu \text{ and } \tilde{T}_{\mu\nu} = \tilde{a}_{\mu,\nu} - \tilde{\Gamma}^\alpha_{\ \mu\nu} a_\alpha \text{ in } \tilde{x}^\mu, \tag{1.15}$$

and works out[8] the transformation properties of the new coordinate dependent quantity Γ, one finds

$$\tilde{\Gamma}^\alpha_{\ \mu\nu} = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\frac{\partial x^\gamma}{\partial \tilde{x}^\mu}\frac{\partial x^\delta}{\partial \tilde{x}^\nu}\Gamma^\beta_{\ \gamma\delta} + \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\frac{\partial^2 x^\beta}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}. \tag{1.16}$$

[8] Exercise: Work out the transformation properties of the coordinate dependent quantity $\Gamma^\alpha_{\mu\nu}$.