Foreword

This material was prepared by D. Puetzfeld as part of Iowa States Astro 405/505 (Fall 2004) course. Comments and suggestions are welcome! Please report any errors, typos, etc. (probably many) back to me (dpuetz@iastate.edu). You can find the material covered in the single lectures online under www.thp.uni-koeln.de/∼dp (click on teaching). There will also be some small computer algebra programs available on this site. The script only supplements the lecture, it is not a substitute for attending the lecture!

Warning: This is a "living" script, i.e. I will update it occasionally without notice.

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The quantity $\Gamma^{\alpha}_{\mu\nu}$, which in general has n^3 independent components, is called an *affine connection*[9]. The following properties of connections follow from (1.16): (i) The difference of two connections ${}^1\Gamma^{\alpha}_{\mu\nu} - {}^2\Gamma^{\alpha}_{\mu\nu}$ is a tensor of the order $(1,2)$. (ii) If $\Gamma^{\alpha}_{\mu\nu}$ is a connection then $\Gamma^{\alpha}_{\nu\mu}$ is also a connection. (iii) $\Gamma^{\alpha}_{(\mu\nu)}$ is a symmetric connection. (iv) $\Gamma^{\alpha}_{[\mu\nu]}$ is a tensor called *torsion.* (v) A general connection can be split up into a symmetric connection and a tensor $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{(\mu\nu)} + \Gamma^{\alpha}_{[\mu\nu]}$. (vi) It is always possible to find a coordinate transformation $x^\mu \to \tilde{x}^\mu$ at a given point P such that $\tilde{\Gamma}^{\alpha}_{\mu\nu}\Big|_P = 0$. (vii) Stronger version of the last statement: Given an arbitrary curve L we can always introduce coordinates in which $\tilde{\Gamma}^{\alpha}_{\mu\nu}\Big|_L = 0$.

In General Relativity we will be concerned only with symmetric connections[10]. The tensor in (1.15) is usually called *covariant derivative* (in this case of a $(0,1)$ tensor, i.e. covariant vector field), we define[11]

$$D_\nu a_\mu \equiv a_{\mu;\nu} := a_{\mu,\nu} - \Gamma^{\alpha}_{\mu\nu} a_\alpha.$$

Rules for covariant differentiation: (i) $\phi_{;\alpha} \equiv \phi_{,\alpha}$ for a scalar ϕ. (ii) $(A^{\cdots}{}_{\ldots} B^{\cdots}{}_{\ldots})_{;\alpha} = A^{\cdots}{}_{\ldots;\alpha} B^{\cdots}{}_{\ldots} + A^{\cdots}{}_{\ldots} B^{\cdots}{}_{\ldots;\alpha}$. For a tensor of rank (k,l) we have[12][13]

$$\begin{aligned} T^{\mu\nu\ldots}{}_{\alpha\beta\ldots;\gamma} = T^{\mu\nu\ldots}{}_{\alpha\beta\ldots,\gamma} \quad & + \Gamma^{\mu}_{\lambda\gamma} T^{\lambda\nu\ldots}{}_{\alpha\beta\ldots} + \Gamma^{\nu}_{\lambda\gamma} T^{\mu\lambda\ldots}{}_{\alpha\beta\ldots} + \ldots \\ & - \Gamma^{\lambda}_{\alpha\gamma} T^{\mu\nu\ldots}{}_{\lambda\beta\ldots} - \Gamma^{\lambda}_{\beta\gamma} T^{\mu\nu\ldots}{}_{\alpha\lambda\ldots} + \ldots . \end{aligned} \tag{1.17}$$

1.1.3 Autoparallels

We learned that the connection $\Gamma^{\alpha}_{\mu\nu}$ allows us to define the covariant derivative $a_{\mu;\nu}$ of a vector which is a tensor. Thus, we can transport a vector from a point A to a point B, the vector at point B has to be considered as the equivalent to the vector at A. We will call this operation the *parallel transport* defined by a connection $\Gamma^{\alpha}_{\mu\nu}$. In general the parallel transport between two points will depend on the path taken. Now let us consider a curve L in n dimensions given by $x^\mu = f^\mu(\lambda)$, with λ being a scalar parameter. This curve shall connect the two points A and B, additionally a^μ shall be

[9]Remark: Note that (1.15) will also be a tensor in the case $\Gamma^{\alpha}_{\mu\nu} \neq \Gamma^{\alpha}_{\nu\mu}$. Quote: " [...] the essential achievement of General Relativity, namely to overcome "ridgid" space (i.e. the inertial frame), is only indirectly connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the "displacement field" ($\Gamma^{\alpha}_{\beta\gamma}$), which expresses the infinitesimal displacement of vectors. It is this which replaces the parallelism of spatially arbitrarily separated vectors fixed by the inertial frame (i.e. the equality of corresponding components) by an infinitesimal operation. This makes it possible to construct tensors by differentiation and hence to dispense with the introduction of "ridgid" space (the inertial frame). In the face of this, it seems to be of secondary importance in some sense that some particular Γ field can be deduced from a Riemannian metric [...]", A. Einstein (1955), translation by F. Gronwald, D. Hartley, F.W. Hehl.

[10]Exercise: Show that in this case the connection has $n^2(n+1)/2$ independent components.

[11]Note: In case of a non-symmetric connection there is some ambiguity at this point.

[12]Exercise: Show that the Kronecker tensor is covariantly constant, i.e. $\delta^{\mu}{}_{\nu;\alpha} = 0$.

[13]Exercise: Show that $a_{[\mu;\nu]} = a_{[\mu,\nu]} - \Gamma^{\alpha}_{[\mu\nu]} a_\alpha$.

a vector given at A. If we parallel transport a^μ along L we obtain another vector at B which we call b^μ. We assume that a^μ is tangent to L in A, i.e. $\alpha^\mu = \frac{dx^\mu}{d\lambda}\Big|_A$. If we transport a^μ along L to B the final result b^μ will not necessarily be tangent to the curve. The special case in which the transported vector b^μ is tangent to the curve at every point of L will be called a geodesic curve or simply *autoparallel*. The condition which characterizes a autoparallel curve is

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = f(\lambda)\frac{dx^\mu}{d\lambda}, \tag{1.18}$$

which has to be satisfied at every point of the curve[14]. Note that the solution of this equation will be completely determined by the point A and the direction of the tangent vector at A. If we introduce another curve parameter σ, reparametrize the curve with the help of this parameter $\lambda = \lambda(\sigma)$, and choose the new parameter in such a way that

$$\frac{d^2\sigma}{d\lambda^2} = f(\lambda)\frac{d\sigma}{d\lambda},$$

then the autoparallel equation reduces to

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} = 0,$$

and we call σ an *affine parameter* of the geodesic.

1.1.4 Curvature

From the connection we can construct another tensor which is called the *curvature tensor* of the space. One can obtain its definition by taking the antisymmetric part

[14]For those of you who do not like this notation: If we have a manifold M, we may define the parallel transport of a vector along the curve. Let $c :]a,b[\rightarrow M$ be the curve on M, its image (for simplicity) shall be covered by a single chart (U,ϕ) whose coordinate is $x = \phi(p)$. Let X be a vector field defined along $c(t)$,

$$X|_{c(t)} = X^\mu(ct)\, e_\mu|_{c(t)} = X^\mu(ct)\frac{\partial}{\partial x^\mu}\Big|_{c(t)}.$$

If X satisfies the condition

$$\nabla_V X = 0 \text{ for any } t \in]a,b[,$$

X is said to be parallel transported along $c(t)$. Here $V = \frac{d}{dt} = \frac{dx^\mu(c(t))}{dt}e_\mu\Big|_{c(t)}$ is the tangent vector to $c(t)$. If the tangent vector itself is parallel transported along $c(t)$, i.e.%

$$\nabla_V V = 0$$

the curve is called an autoparallel. One may also call it the straightest possible line. Note that these curves are also called *geodesics* (we will reserve this term for spaces in which we are able to measure lengths).

of the second covariant derivative of a covariant vector field a_λ, i.e. $a_{\lambda;[\mu\nu]}$[15]. We shall define the *curvature tensor* of rank $(1,3)$ as follows

$$R^\rho{}_{\lambda\mu\nu} = -\Gamma^\rho_{\lambda\mu,\nu} + \Gamma^\rho_{\lambda\nu,\mu} - \Gamma^\sigma_{\lambda\mu}\Gamma^\rho_{\sigma\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\rho_{\sigma\mu}. \tag{1.19}$$

Hence, the connection completely determines the curvature tensor. Properties: (i) $R^\rho{}_{\lambda\mu\nu} = -R^\rho{}_{\lambda\nu\mu}$ (ii) If the connection is symmetric $R^\rho{}_{[\lambda\mu\nu]} = 0$. (iii) If the connection is symmetric and $R^\rho{}_{\lambda\mu\nu}$ vanishes in a region M, then it is possible to obtain $\Gamma^\lambda_{\mu\nu} = 0$ in M by an appropriate coordinate transformation. (iv) In case $R^\rho{}_{\lambda\mu\nu}$ vanishes in a region M than the parallel transport between two points along curves which lie entirely in M is path independent.

1.1.5 Metric & Riemannian space

Up to this point our introduction was fairly general and did not allow us to measure distances in our space. We will now switch over to a *metric space*, which is a space in which it is possible to define a scalar distance for each pair of neighboring points. There are many different examples for metric spaces: (i) The Euclidean space (in Cartesian coordinates X^α) with $d\sigma^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2$ or (ii) the Minkowski space (in an inertial frame X^α) with $ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2$. If we introduce general coordinates x^μ by $X^\mu = f^\mu(x^\nu)$ we have $dX^\mu = \frac{\partial X^\mu}{\partial x^\alpha} dx^\alpha$. Within such coordinates the above line elements will be of the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1.20}$$

i.e. homogeneous and quadratic in the dx^μ, with some symmetric quantity $g_{\mu\nu} = g_{(\mu\nu)}$. The relation in (1.20) characterizes a Riemannian space[16]. In general the components of tensor $g_{\mu\nu}$ are arbitrarily given functions of the coordinates, and therefore it is not possible to reduce them by a coordinate transformation to the simple form as in the Euclidean or Minkowski space, both of which are special cases of Riemannian space.

[15]Exercise: Show that

$$a_{\lambda;[\mu\nu]} = R^\rho{}_{\lambda\mu\nu} a_\rho - 2\Gamma^\rho{}_{[\mu\nu]} a_{\lambda;\rho},$$

with

$$R^\rho{}_{\lambda\mu\nu} = -\Gamma^\rho_{\lambda\mu,\nu} + \Gamma^\rho_{\lambda\nu,\mu} - \Gamma^\sigma_{\lambda\mu}\Gamma^\rho_{\sigma\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\rho_{\sigma\mu}.$$

[16]A metric is called $\left\{\begin{array}{l}\text{positive-}\\ \text{negative-}\\ \text{in-}\end{array}\right\}$definite if $\left\{\begin{array}{l}X^2 > 0\\ X^2 < 0\\ \text{else}\end{array}\right\}$ for all vectors X^μ. Of course the metric allows us to measure angles in the usual way. For two vector X^μ and Y^μ with $X^2 \neq 0$ and $Y^2 \neq 0$ we have

$$\cos(X,Y) = \frac{g_{\mu\nu} X^\mu Y^\nu}{\sqrt{|g_{\alpha\beta} X^\alpha X^\beta|}\sqrt{|g_{\lambda\sigma} Y^\lambda Y^\sigma|}}.$$