Foreword

This material was prepared by D. Puetzfeld as part of Iowa States Astro 405/505 (Fall 2004) course. Comments and suggestions are welcome! Please report any errors, typos, etc. (probably many) back to me (dpuetz@iastate.edu). You can find the material covered in the single lectures online under www.thp.uni-koeln.de/∼dp (click on teaching). There will also be some small computer algebra programs available on this site. The script only supplements the lecture, it is not a substitute for attending the lecture!

Warning: This is a "living" script, i.e. I will update it occasionally without notice.

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The tensor $g_{\mu\nu}$ of rank $(0,2)$ is called *metric*. The metric tensor allows us to construct a scalar from two infinitesimal displacements dx_1^μ and dx_2^ν at the some point A, i.e. $g_{\mu\nu}dx_1^\mu dx_2^\nu$. This is a direct generalization of the *scalar product* of two vectors as we know it from Euclidean space. Note the difference to a space without metric where we could only form scalars from two vectors if one vector is contravariant and the other covariant. Our ability to form the product of two covariant or contravariant quantities allows us to define the co-/contravariant equivalent to a contra-/covariant quantity, i.e. to raise and lower the indices of quantities. Examples:

$$a_\mu = g_{\mu\nu}a^\nu, \quad T^\nu{}_\mu = g_{\mu\alpha}T^{\nu\alpha}, \quad T_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}, \ldots .$$

The contravariant form of the metric is defined via $g_{\mu\alpha}g^{\nu\alpha} = \delta_\mu^\nu$. Two important theorems with respect to the metric are: (i) At a given point it is always possible to find a coordinate transformation $x^\mu \rightarrow \tilde{x}^\mu$ such that $\tilde{g}_{\mu\nu} = \begin{cases} \pm 1 & \text{for} \quad \mu = \nu \\ 0 & \text{for} \quad \mu \neq \nu \end{cases}$. (ii) If we consider only real transformations the number of minus and plus signs in the diagonal form of the metric does not change. Since we now know what a metric looks like we can also work out the form of a connection in a Riemannian space. Under the assumption that the parallel transport of a vector does not change its length one obtains

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(g_{\beta\nu,\mu} + g_{\mu\beta,\nu} - g_{\mu\nu,\beta}\right). \tag{1.21}$$

The quantities in (1.21) are usually called *Christoffel symbols*. This form of the connection is exclusively used in Riemannian geometry and is completely determined by the metric[17]. Let us now come back to the geodesic equation and its form in a Riemannian space. Of course the connection in (1.18) is now the connection from (1.21). Since we are now able to measure distances it is rather natural to use the proper length $s = \int_A^B ds$ of a curve as parameter λ. With $\lambda = s$ we shall have $\frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{ds} \equiv u^\mu$, and the tangent vector u^μ is normalized via: $g_{\mu\nu}u^\mu u^\nu = 1$. The autoparallel equation (1.18) takes the form

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = f\left(s\right)u^\mu.$$

It can be shown that s is an affine parameter, therefore we end up with

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0. \tag{1.22}$$

This equation is called *geodesic equation*. Furthermore one can proof that geodesics in a Riemannian space are either the curves of maximal or minimal length connecting the points A and B. Since we already know the general definition of the curvature tensor we can now define the *Riemann tensor* which is nothing else than (1.19) together with the symmetric connection from (1.21). The following symmetries hold: $R^\rho{}_{\lambda\mu\nu} = -R^\rho{}_{\lambda\nu\mu}$,

[17]Exercise: Show that $g_{\mu\nu;\sigma} = 0$.

$R^{\rho}{}_{[\lambda\mu\nu]} = 0$, $R^{\sigma}{}_{\mu[\nu\alpha;\beta]} = 0$[18], and[19] $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} = R_{\alpha\beta\mu\nu}$. Thus, in a 4 dimensional space the Riemann tensor has 20 independent components. The contraction $R^{\alpha}{}_{\alpha\mu\nu}$ vanishes identically. The only non-vanishing contraction is the *Ricci tensor*[20]

$$R_{\mu\nu} := R^{\alpha}{}_{\mu\nu\alpha},$$

which (in a 4 dimensional spacetime) has 10 independent components. The contraction of the Ricci tensor

$$R := R^{\mu}{}_{\mu}$$

is called the *Ricci scalar*, the combination of the Ricci tensor and scalar

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

is called the *Einstein tensor*. The covariant divergence of the Einstein tensor vanishes, i.e. $G^{\mu}{}_{\nu;\mu} = 0$.

Weyl-Tensor Another important quantity with respect to the classification of different spacetimes is the so-called *Weyl tensor*. We briefly mention some of its properties here. The Riemann tensor may be expressed (in 4 dimensions) by trace-free tensor quantities in the following way

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= C_{\mu\nu\alpha\beta} + \frac{1}{2}\left(g_{\mu\beta}L_{\nu\alpha} + g_{\beta\alpha}L_{\mu\beta} - g_{\mu\alpha}L_{\nu\beta} - g_{\nu\beta}L_{\mu\alpha}\right) \\ &\quad + \frac{1}{12} R \left(g_{\mu\beta}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\beta}\right), \end{aligned} \tag{1.23}$$

with $L_{\mu\nu} := R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R$, which satisfies $L^{\mu}{}_{\mu}$. The tensor $C_{\mu\nu\alpha\beta}$ with $C^{\alpha}{}_{\mu\nu\alpha} = 0$ is defined by (1.23) and called the Weyl tensor or conformal curvature tensor (because $\tilde{C}^{\mu}{}_{\nu\alpha\beta} = C^{\mu}{}_{\nu\alpha\beta}$ under conformal transformations $\tilde{g}_{\mu\nu} = \phi g_{\mu\nu}$). It has the same (in addition to $C^{\alpha}{}_{\mu\nu\alpha} = 0$) symmetries as the Riemann tensor $C_{\mu\nu\alpha\beta} = -C_{\nu\mu\alpha\beta} = -C_{\mu\nu\beta\alpha} = C_{\alpha\beta\mu\nu}$, $C_{\mu[\nu\alpha\beta]} = 0$, and therefore 10 independent components.

Isometries Without giving a derivation at this point, we notice that the condition for the existence of isometric mappings is the existence of solutions X^{μ} of the equation

$$X_{\mu;\nu} + X_{\nu;\mu} = 0. \tag{1.24}$$

This equation is called *Killing equation* and vectors X^{μ} which satisfy it are called *Killing vectors*. The existence of a Killing vector expresses a certain intrinsic symmetry property of the space.

[18] This is the *Bianchi identity*.

[19] Remember that we can lower and raise the indices with the metric.

[20] Exercise: Show that the Ricci tensor is symmetric $R_{\mu\nu} = R_{\nu\mu}$.

1.2 Part II - From Newton to Einstein

Goal: Sketch ideas which led to the formulation of GR.

1.2.1 Newton's gravitational theory

Newton's theory of gravitation has been very successful, think of the detailed study of the motion of the planets, e.g. According to Newton the gravitational force between two bodies of mass m_1 and m_2 placed at the positions $\mathbf{r}_{1,2}$ is given by

$$\mathbf{F}_{21} = \frac{Gm_1m_2}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|} = -\mathbf{F}_{12},$$

here G denotes Newton's gravitational constant[21], and the vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ points from m_1 to m_2. Lets us now distinguish the mass $m_2 = M$ as a field-generating gravitational mass and $m_1 = m$ as a test mass in the field of m_2. We introduce a gravitational field describing the force per unit mass $\mathbf{f} := \frac{\mathbf{F}}{m} = \frac{GM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}$, such that $\mathbf{F}_{Mm} = m\mathbf{f}$. This field may be expressed by a potential $\phi = -G\frac{M}{|\mathbf{r}|}$, i.e. $\mathbf{f} = -\nabla\phi$. By assumption the mass generating the gravitational field is M, hence $\nabla\mathbf{f} = 4\pi GM\delta^3(\mathbf{r})$. Therefore, if we replace the pointmass by a continuous matter distribution we obtain the field equation for the gravitational potential $\nabla^2\phi = -4\pi G\rho(\mathbf{r})$. In summary the characteristic properties of Newton's gravitational theory are: (i) It is a scalar theory (i.e. has a scalar potential ϕ and therefore a scalar source of the field, the source being the mass-density of the material distribution). (ii) The field equation is a linear partial differential equation of second order. (iii) The theory uses the pre-relativistic concepts of absolute space and absolute time, the field ϕ has no dynamic properties. Consequently Newton's theory represents an action at a distance theory. Nevertheless Newtonian gravity is a successful theory on certain length and time scales, a search for a new relativistic theory of gravitation should therefore be guided by the demand for an appropriate limit in which it reduces to Newton's theory.

1.2.2 How to formulate a relativistic gravity theory

A relativistic generalization of Newtonian gravity should at least make use of the spacetime concepts which we already know from *special relativity.* In special relativity continuous matter is described by the symmetric *stress-energy-momentum tensor* $T_{\mu\nu} = T_{(\mu\nu)}$. Example: Maxwell's stress-energy-momentum tensor for the electromagnetic field[22] ${}^{\text{Max}}T_{\mu\nu}$, which has the general structure[23]

$${}^{\text{Max}}T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{0a} \\ T_{a0} & T_{ab} \end{pmatrix} = \begin{pmatrix} \text{energy density} & \text{momentum density} \\ \text{energy flux density} & \text{momentum flux density} \end{pmatrix}.$$

[21] In SI units we have $G = 6.673(10) \times 10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}$.

[22] ${}^{\text{Max}}T_{\alpha\beta} = -F_{\alpha\mu}F_\beta{}^\mu + \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}$.

[23] Latin indices shall run from $a, b = 1, \ldots, 3$. Momentum flux density $\equiv$ stress.