Foreword

This material was prepared by D. Puetzfeld as part of Iowa States Astro 405/505 (Fall 2004) course. Comments and suggestions are welcome! Please report any errors, typos, etc. (probably many) back to me (dpuetz@iastate.edu). You can find the material covered in the single lectures online under www.thp.uni-koeln.de/∼dp (click on teaching). There will also be some small computer algebra programs available on this site. The script only supplements the lecture, it is not a substitute for attending the lecture!

Warning: This is a "living" script, i.e. I will update it occasionally without notice.

Last update: 9th December 2004

We can of course go ahead and mimic this structure as source for the gravitational field. Consequently, T_{00} would represent the material energy density, T_{0a} and T_{a0} the momentum and energy flux density of matter, and T_{ab} will be the mechanical stresses acting in the continuous matter which we want to consider here. Let us now try to construct a gravity theory in which $T_{\mu\nu}$ somehow represents the source of the gravitational field. An obvious choice with respect to the structure of Newton's theory would be to construct a scalar theory of gravitational interaction from the stress-energy-momentum tensor $T_{\mu\nu}$. In fact this was done a few years before the formulation of General Relativity in Minkowski spacetime[24] by using the scalar $T = \eta^{\alpha\beta}T_{\alpha\beta}$ and the field equation for the gravitational potential given by

$$\Box\phi \equiv \partial_\alpha \eta^{\alpha\beta}\partial_\beta\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = \kappa T,$$

where κ denotes a coupling constant. This is a truly relativistic gravitational theory which reduces to Newtonian gravity in the first approximation. Unfortunately, it turned out that this theory is in disagreement with observations[25]. Since it is not possible to construct a vector from the tensor $T_{\mu\nu}$ in a straightforward way[26] it is rather natural to consider a tensorial theory as a candidate for a relativistic gravity theory. In such a theory all 10 components of the tensor $T_{\mu\nu}$ will act as sources for the gravitational field, therefore the corresponding potential should also by a symmetric tensor of rank $(0, 2)$. General Relativity will turn out to be a tensorial theory of the gravitational field, the flat Minkowski background in General Relativity will be replaced by a curved Riemannian spacetime. The metric tensor $g_{\mu\nu}$ of the Riemannian space will play the role of the gravitational potential.

The equivalence principle During the development of General Relativity the equivalence principle played an important role. The equivalence principle as formulated by Einstein is a generalization of the assumption, which was already used in Newton's gravitational theory, that the gravitational mass of body and the inertial mass of a body are the same[27]. The experimental verification[28] of the equivalence of these a priori different mass concepts is embodied in the following *principle of equivalence*[29]: Gravitational and inertial forces are completely equivalent, i.e. they are of an identical nature and consequently it is impossible to separate them by any physical experiment. This principle

[24] We will use the symbol $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ for the metric in Minkowski space.

[25] Wrong prediction for the rotation of the perihelion of mercury. No deflection of light, since a scalar field cannot be coupled reasonably to the energy-momentum tensor of the Maxwell field, which is traceless.

[26] Nevertheless there have been several attempts to formulate such a theory.

[27] $\mathbf{F} = m_{\text{inert}}\ddot{\mathbf{r}} = m_{\text{grav}}\mathbf{f} = \mathbf{F}_{Mm}$.

[28] $(m_{\text{grav}} - m_{\text{inertl}})/m_{\text{gr}} < 10^{-8}$ (Eötvös 1922 et al.) $< 10^{-11}$(Dicke et al. 1964) $< 10^{-12}$ (Braginski et al. 1971). A future satellite experiment (Worden et al. 1978) is targeted to achieve $< 10^{-15} - 10^{-18}$.

[29] In the literature there is usually a distinction between the so-called weak (WEP) and strong (SEP)/Einstein (EEP) equivalence principle. By using the term WEP one ususually confines the validity of the EP to mechanical systems, therefore it is the same principle as already encountered in Newton's theory. In case of the SEP all laws of nature are subject to the EP.

has some important consequences: (i) Gravitational forces (or equivalently accelerations) have to be described in the same way as inertial forces (accelerations). In an inertial frame a particle moves on a straight line described by $\frac{d^2x^\mu}{ds^2} = 0$. We already know that in a general frame this equation will look like eq. (1.22). The second term stemming from the use of a non-inertial frame. Therefore the Christoffel symbols generally describe the inertial accelerations. The EP now tells us that the gravitational accelerations are also described by the Christoffel symbols or, more precisely, the Christoffel symbols describe the sum of the inertial and gravitational accelerations[30]. Since the Christoffel symbols are derived from the metric tensor we conclude that the metric will play the role of the gravitational potential. (ii) In the general case of gravitational accelerations we cannot make the Christoffel symbols vanish everywhere and therefore space cannot be the flat Minkowski space any longer[31]. Consequently, the gravitational field will be represented by the fact that the spacetime is curved. (iii) An immediate consequence of the last two remarks is that there are no global inertial frames in General Relativity or, stated in another way, acceleration has no longer an absolute meaning in GR. There are several physical consequences which can be deduced from the validity of the EP, e.g.: (i) Necessity of the deflection of a light ray in a gravitational field. (ii) Gravitational redshift.

1.2.3 The field equations of General Relativity

Now let us come back to the field equations of a relativistic formulation of a gravity theory. We already mentioned the energy-momentum tensor as source for the gravitational field, since the scalar potential ansatz was not successful and the construction of a vectorial theory is not straightforward, it is rather natural to consider the tensorial case[32]. Hence the field equations should be of the form $F_{\mu\nu} = \kappa T_{\mu\nu}$ (with κ being a constant which has to be specified), where we have to construct the tensor on the lhs from the gravitational potential. As we saw before the role of the gravitational potential is now played by the metric. Since we want to retain the Newtonian limit we assume that $F_{\mu\nu}$ contains only derivatives of $g_{\mu\nu}$ up to the second order. In Riemannian geometry the only tensors which can be constructed from the metric $g_{\mu\nu}$ and its first and second derivatives which is linear in the second derivatives are the Riemann tensor and its contractions and the tensor $g_{\mu\nu}$ itself. Therefore an appropriate ansatz for the lhs is $F_{\mu\nu} = AR_{\mu\nu} + Bg_{\mu\nu}R + Cg_{\mu\nu}$, where A, B, and C are some constants. The requirement that the divergence (in general coordinates) of the energy-momentum vanishes, i.e. $T^{\mu\nu}{}_{;\nu} = 0$, determines[33] the constants A and B in the ansatz for $F_{\mu\nu}$, namely $A = 1$

[30]Note that there will be no way to split this sum unambiguously into two terms representing inertial and gravitational accelerations.

[31]Note that we can in principle distinguish between an accelerated observer and the gravitational field of a point mass in a non-local experiment. In case of an experiment in a gravitational field we would look for tidal effects by comparing the trajectories of two test-masses.

[32]Note that we already learned that a frame independent form of the field equations is mandatory since there are no longer any preferred coordinates.

[33]Remember our results and the definition of $G_{\mu\nu}$ in the previous section.

and $B = \frac{1}{2}$. Renaming $C \equiv \Lambda$ for historical reasons[34] we end up with the field equations of General Relativity

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + g_{\alpha\beta}\Lambda \equiv G_{\mu\nu} + g_{\alpha\beta}\Lambda = \kappa T_{\alpha\beta}. \tag{1.25}$$

For $\Lambda = 0$ the vacuum ($T_{\mu\nu} = 0$) field equations reduce to $R_{\alpha\beta} = 0$, since $R = -\kappa T^{\alpha}{}_{\alpha} \Leftrightarrow R_{\mu\nu} = \kappa \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T^{\mu}{}_{\mu}\right)$[35]. The coupling constant κ in the field equations is determined by demanding that (1.25) reduces to the Poisson equation in the case of weak static gravitational fields $g_{00} \simeq \left(1 + \frac{2\phi}{c^2}\right)$ and non-relativistic matter $T_{00} \sim \rho$, i.e. $\nabla^2 g_{00} \sim \frac{8\pi G}{c^4} T_{00}$, therefore $\kappa = 8\pi G/c^4$.

1.3 Part III - The Schwarzschild solution

Goal: Discuss one exact solution in GR, and derive its physical consequences.

1.3.1 Symmetry considerations

Let us now discuss one simple exact solution of the field equations of General Relativity (1.25) without cosmological constant, i.e. $G_{\mu\nu} = \kappa T_{\mu\nu}$. The solution to be discussed was found by astronomer K. Schwarzschild (1916) only a few months after Einstein published his new gravitational theory. Although the solution is simple it describes most of the general relativistic effects in the planetary system[36]. The solution found by Schwarzschild describes the gravitational field of a spherically symmetric body. We will only be concerned with the field outside of the body ($T_{\mu\nu} = 0$), hence the field equations reduce to $R_{\mu\nu} = 0$, as shown in the last section.

1.3.2 The basic form of the line element

Starting from the spherically symmetric line element[37]

$$ds^2 = e^{\nu(t,r)} c^2 dt^2 - e^{\mu(t,r)} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{1.26}$$

Here $\nu(t, r)$ and $\mu(t, r)$ represent arbitrary functions of the coordinates. Since we want to determine the gravitational field of a spherically symmetric body at rest we demand

[34] This term is usually called cosmological constant.

[35] Exercise: How do the vacuum field equations look like if we have a non-vanishing cosmological constant. Note that in this case there is problem with the Newtonian limit of the field equations.

[36] In fact on first sight one would not expect a simple and physically meaningful solution of the general field equations.

[37] This general form of the line element can by found by means of the Killing equation (1.24). In general a spacetime is called *spherically symmetric* if there are three linear independent spacelike (${}^{n}X^{\mu}\,{}^{n}X_{\mu} < 0,\ n = 1, \ldots, 3$) Killing vectorfields $\left({}^1X,\ {}^2X,\ {}^3X\right)$ with: $\left[{}^1X, {}^2X\right] = {}^3X$, $\left[{}^2X, {}^3X\right] = {}^1X$, and $\left[{}^3X, {}^1X\right] = {}^2X$.