An exact plane-fronted wave solution in metric-affine gravity

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Abstract

We study plane–fronted electrovacuum waves in metric–affine gravity (MAG) with cosmological constant in the triplet ansatz sector of the theory. Their field strengths are, on the gravitational side, curvature $R_\alpha{}^\beta$, nonmetricity $Q_{\alpha\beta}$, torsion T^α and, on the matter side, the electromagnetic field strength F. Here we basically present, after a short introduction into MAG and its triplet subcase, the results of earlier joint work with García, Macías, and Socorro [1]. Our solution is based on an exact solution of Ozsváth, Robinson, and Rózga describing type N gravitational fields in general relativity as coupled to electromagnetic null-fields.

1 Introduction

Metric-affine gravity (MAG) represents a gauge theoretical formulation of a theory of gravity which, in contrast to general relativity theory (GR), is no longer confined to a pseudo-Riemannian spacetime structure (cf. [2]). There are new geometric quantities emerging in this theory, namely torsion and nonmetricity, which act as additional field strengths comparable to curvature in the general relativistic case. Due to this general ansatz, several alternative gravity theories are included in MAG, the Einstein-Cartan theory e. g., in which the nonmetricity vanishes and the only surviving post-Riemannian quantity is given by the torsion. One expects that the MAG provides the correct description for early stages of the universe, i. e. at high energies at which the general relativistic description is expected to break down. In case of vanishing post-Riemannian quantities, MAG proves to be compatible with GR. In contrast to GR, there are presently only a few exact solutions known in MAG (cf. [3]), what could be ascribed to the complexity of this theory.

In the following we will give a short overview of the field equations of MAG and the geometric quantities featuring therein. Especially, we will present the results of the work of Obukhov et al. [4] who found that a special case of MAG,

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the so called triplet ansatz, is effectively equivalent to an Einstein-Proca theory. Within the framework of this ansatz, we will show how one is able to construct a solution of the MAG field equations on the basis of a plane-fronted wave solution of GR which was originally presented by Ozsváth et al. in [5].

2 MAG in general

In MAG we have the metric $g_{\alpha\beta}$, the coframe ϑ^α, and the connection 1-form $\Gamma_\alpha{}^\beta$ [with values in the Lie algebra of the four-dimensional linear group $GL(4,R)$] as new independent field variables. Here $\alpha, \beta, \ldots = 0,1,2,3$ denote (anholonomic) frame indices. Spacetime is described by a metric-affine geometry with the gravitational field strengths nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, torsion $T^\alpha := D\vartheta^\alpha$, and curvature $R_\alpha{}^\beta := d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta$. A Lagrangian formalism for a matter field Ψ minimally coupled to the gravitational potentials $g_{\alpha\beta}$, ϑ^α, $\Gamma_\alpha{}^\beta$ has been set up in [2]. The dynamics of this theory is specified by a total Lagrangian

$$L = V_{\rm MAG}(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha{}^\beta) + L_{\rm mat}(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi). \tag{1}$$

The variation of the action with respect to the independent matter field and the gauge potentials leads to the field equations:

$$\begin{aligned} \frac{\delta L_{\rm mat}}{\delta \Psi} &= 0, & (2)\\ DM^{\alpha\beta} - m^{\alpha\beta} &= \sigma^{\alpha\beta}, & (3)\\ DH_\alpha - E_\alpha &= \Sigma_\alpha, & (4)\\ DH^\alpha{}_\beta - E^\alpha{}_\beta &= \Delta^\alpha{}_\beta. & (5) \end{aligned}$$

Equation (4) represents the generalized Einstein equation with the energy-momentum 3-form Σ_α as its source whereas (3) and (5) are additional field equations which take into account other aspects of matter, such as spin, shear, and dilation currents represented collectively by the hypermomentum $\Delta^\alpha{}_\beta$. We made use of the definitions of the gauge field excitations,

$$M^{\alpha\beta} := -2\frac{\partial V_{\rm MAG}}{\partial Q_{\alpha\beta}}, \quad H_\alpha := -\frac{\partial V_{\rm MAG}}{\partial T^\alpha}, \quad H^\alpha{}_\beta := -\frac{\partial V_{\rm MAG}}{\partial R_\alpha{}^\beta}, \tag{6}$$

and of the canonical energy-momentum, the metric stress-energy, and the hypermomentum current of the gauge fields,

$$m^{\alpha\beta} := 2\frac{\partial V_{\rm MAG}}{\partial g_{\alpha\beta}}, \quad E_\alpha := \frac{\partial V_{\rm MAG}}{\partial \vartheta^\alpha}, \quad E^\alpha{}_\beta := -\vartheta^\alpha \wedge H_\beta - g_{\beta\gamma} M^{\alpha\gamma}. \tag{7}$$

Moreover, we introduced the canonical energy-momentum, the metric stress-energy, and the hypermomentum currents of the matter fields, respectively,

$$\sigma^{\alpha\beta} := 2\frac{\delta L_{\rm mat}}{\delta g_{\alpha\beta}}, \quad \Sigma_\alpha := \frac{\delta L_{\rm mat}}{\delta \vartheta^\alpha}, \quad \Delta^\alpha{}_\beta := \frac{\delta L_{\rm mat}}{\delta \Gamma_\alpha{}^\beta}. \tag{8}$$

Provided the matter equation (2) is fulfilled, the following Noether identities hold:

$$D\Sigma_\alpha = \left(e_\alpha\rfloor T^\beta\right)\wedge\Sigma_\beta - \frac{1}{2}\left(e_\alpha\rfloor Q_{\beta\gamma}\right)\sigma^{\beta\gamma} + \left(e_\alpha\rfloor R_\beta{}^\gamma\right)\wedge\Delta^\beta{}_\gamma, \qquad (9)$$

$$D\Delta^\alpha{}_\beta = g_{\beta\gamma}\sigma^{\alpha\gamma} - \vartheta^\alpha\wedge\Sigma_\beta. \qquad (10)$$

They show that the field equation (3) is redundant. Thus we only need to take into account (4) and (5). As suggested in [2], the most general parity conserving Lagrangian expressed in terms of the irreducible pieces (cf. [2]) of nonmetricity $Q_{\alpha\beta}$, torsion T^α, and curvature $R_{\alpha\beta}$ reads

$$\begin{aligned} V_{\rm MAG} = \; & \frac{1}{2\kappa}\Big[-a_0\,R^{\alpha\beta}\wedge\eta_{\alpha\beta} - 2\lambda\eta + T^\alpha\wedge{}^\star\Big(\sum_{I=1}^{3}a_I\,{}^{(I)}T_\alpha\Big) \\ & +Q_{\alpha\beta}\wedge{}^\star\Big(\sum_{I=1}^{4}b_I\,{}^{(I)}Q^{\alpha\beta}\Big) + b_5\left({}^{(3)}Q_{\alpha\gamma}\wedge\vartheta^\alpha\right)\wedge{}^\star\left({}^{(4)}Q^{\beta\gamma}\wedge\vartheta_\beta\right) \\ & +2\Big(\sum_{I=2}^{4}c_I\,{}^{(I)}Q_{\alpha\beta}\Big)\wedge\vartheta^\alpha\wedge{}^\star T^\beta\Big] \\ & -\frac{1}{2\rho}R^{\alpha\beta}\wedge{}^\star\Big[\sum_{I=1}^{6}w_I\,{}^{(I)}W_{\alpha\beta} + \sum_{I=1}^{5}z_I\,{}^{(I)}Z_{\alpha\beta} + w_7\,\vartheta_\alpha\wedge\left(e_\gamma\rfloor{}^{(5)}W^\gamma{}_\beta\right) \\ & +z_6\,\vartheta_\gamma\wedge\left(e_\alpha\rfloor{}^{(2)}Z^\gamma{}_\beta\right) + \sum_{I=7}^{9}z_I\,\vartheta_\alpha\wedge\left(e_\gamma\rfloor{}^{(I-4)}Z^\gamma{}_\beta\right)\Big]. \end{aligned} \qquad (11)$$

Note that we decompose the curvature 2-form $R_\alpha{}^\beta$ into its antisymmetric and symmetric parts, i.e. $R_{\alpha\beta} = W_{\alpha\beta} + Z_{\alpha\beta} = R_{[\alpha\beta]} + R_{(\alpha\beta)} \sim$ rotational $\oplus$ strain curvature. The constants entering eq. (11) are the cosmological constant λ, the weak and strong coupling constant κ and ρ, respectively, and the 28 dimensionless parameters

$$a_0,\ldots,a_3, b_1,\ldots,b_5, c_2,\ldots,c_4, w_1,\ldots,w_7, z_1,\ldots,z_9. \qquad (12)$$

We have here the following dimensions: $[\lambda]$ =length^{-2}, $[\kappa]$ =length^2, $[\rho] = [\hbar] = [c] = 1$. The Lagrangian (11) and the presently known exact solutions in MAG have been reviewed in [3].

3 The triplet ansatz

In the following we will briefly review the results of Obukhov et al. [4]. Starting from the most general gauge Lagrangian $V_{\rm MAG}$ in (11), we now investigate the special case with

$$w_1,\ldots,w_7 = 0, \quad z_1,\ldots,z_3, z_5,\ldots,z_9 = 0, \quad z_4 \neq 0. \qquad (13)$$

Thus we consider a general weak part, i.e., we do not impose that one of the weak coupling constants vanishes right from the beginning. However, the strong gravity part of (11) is truncated for simplicity. Its only surviving piece is given by the square of the dilation part of the segmental curvature ${}^{(4)}Z_{\alpha\beta} := \frac{1}{4} g_{\alpha\beta} Z_\gamma{}^\gamma$. In this case, the result of Obukhov et al. [4] reads as follows: Effectively, the curvature $R_{\alpha\beta}$ may be considered as Riemannian, torsion and nonmetricity may be represented by a 1-form ω,

$$Q = k_0\,\omega, \qquad \Lambda = k_1\,\omega, \qquad T = k_2\,\omega, \tag{14}$$

$$T^\alpha = {}^{(2)}T^\alpha = \frac{1}{3}\vartheta^\alpha \wedge T, \tag{15}$$

$$Q_{\alpha\beta} = {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta} = \frac{4}{9}\left(\vartheta_{(\alpha} e_{\beta)} \rfloor \Lambda - \frac{1}{4} g_{\alpha\beta}\Lambda\right) + g_{\alpha\beta} Q. \tag{16}$$

With the aid of the Riemannian curvature $\tilde{R}_{\alpha\beta}$, we denote Riemannian quantities by a tilde, the field equation (4) looks like the Einstein equation with an energy-momentum source that depends on torsion and nonmetricity. Therefore, the field equation (5) becomes a system of differential equations for torsion and nonmetricity alone. In the vacuum case (i.e. $\Sigma_\alpha = 0$ and $\Delta_\alpha{}^\beta = 0$), these differential equations reduce to

$$\frac{a_0}{2}\,\eta_{\alpha\beta\gamma} \wedge \tilde{R}^{\beta\gamma} + \lambda \eta_\alpha = \kappa \Sigma^{(\omega)}_\alpha, \tag{17}$$

$$d\,{}^\star d\omega + m^2\,{}^\star\omega = 0. \tag{18}$$

The four constants m, k_0, k_1, and k_2, which appear in (18) and (14), depend uniquely on the parameters of the MAG Lagrangian (13):

$$\begin{aligned}
k_0 &= 4\,(a_2 - 2a_0)\left(b_3 + \frac{a_0}{8}\right) - 3\,(c_3 + a_0)^2\,,\\
k_1 &= \frac{9}{2}\,(a_2 - 2a_0)\,(b_5 - a_0) - 9\,(c_3 + a_0)\,(c_4 + a_0)\,,\\
k_2 &= 12\left(b_3 + \frac{a_0}{8}\right)(c_4 + a_0) - \frac{9}{2}\,(b_5 - a_0)\,(c_3 + a_0)\,,\\
m^2 &= \frac{1}{z_4\kappa}\left(-4b_4 + \frac{3}{2}a_0 + \frac{k_1}{2k_0}\,(b_5 - a_0) + \frac{k_2}{k_0}\,(c_4 + a_0)\right).
\end{aligned} \tag{19}$$

The energy-momentum source of torsion and nonmetricity $\Sigma^{(\omega)}_\alpha$, which appears in the effective Einstein equation (17), reads

$$\begin{aligned}
\Sigma^{(\omega)}_\alpha = \frac{z_4 k_0^2}{2\rho}\Big\{ &(e_\alpha \rfloor d\omega) \wedge {}^\star d\omega - (e_\alpha \rfloor {}^\star d\omega) \wedge d\omega \\
&+ m^2\left[(e_\alpha \rfloor \omega) \wedge {}^\star\omega + (e_\alpha \rfloor {}^\star\omega) \wedge \omega\right]\Big\}.
\end{aligned} \tag{20}$$

This energy-momentum is exactly that of a Proca 1-form field. The parameter m in (18) has the meaning of the mass parameter ($[m]$ =length^{-1}). If m vanishes, the constrained MAG theory looks similar to the Einstein-Maxwell theory, as can be seen immediately by comparing (20) with the energy-momentum current of the Maxwell theory

$$\Sigma_\alpha^{\rm Max} = \frac{1}{2}\Big\{(e_\alpha\rfloor dA)\wedge {}^\star dA - (e_\alpha\rfloor {}^\star dA)\wedge dA\Big\}, \tag{21}$$

where A denotes the electromagnetic potential 1-form ($F = dA$). Note that $m = 0$ leads to an additional constraint among the coupling constants (cf. eq. (19)).

4 Plane-fronted waves in GR

Ozsváth, Robinson and Rózga [5] dealt with a solution of the Einstein-Maxwell equations. Here we sketch their procedure in order to show how to generalize it to the triplet subcase of MAG. Since we perform our calculations with arbitrary gravitational coupling constant κ, the results presented here differ by some factor of κ from the original ones in [5]. Using the coordinates $(\rho,\sigma,\zeta,\bar\zeta)$, we start with the line element

$$ds^2 = 2(\vartheta^{\hat 0}\otimes\vartheta^{\hat 1} + \vartheta^{\hat 2}\otimes\vartheta^{\hat 3}), \tag{22}$$

and the coframe (the bar denotes complex conjugation)

$$\vartheta^{\hat 0} = \frac{1}{p}\,d\zeta, \qquad \vartheta^{\hat 1} = \frac{1}{p}\,d\bar\zeta, \qquad \vartheta^{\hat 2} = -d\sigma, \qquad \vartheta^{\hat 3} = \left(\frac{q}{p}\right)^2 (s\,d\sigma + d\rho)\,. \tag{23}$$

In order to write down the coframe in a compact form, we made use of the abbreviations p, q, and s which are defined in the following way:

$$p(\zeta,\bar\zeta) = 1 + \frac{\lambda}{6}\,\zeta\bar\zeta, \tag{24}$$

$$q(\sigma,\zeta,\bar\zeta) = \left(1 - \frac{\lambda}{6}\,\zeta\bar\zeta\right)\alpha(\sigma) + \zeta\bar\beta(\sigma) + \bar\zeta\beta(\sigma), \tag{25}$$

$$\begin{aligned} s(\rho,\sigma,\zeta,\bar\zeta) &= -\frac{\rho^2\lambda}{6}\,\alpha^2(\sigma) - \rho^2\beta(\sigma)\,\bar\beta(\sigma) + \rho\,\partial_\sigma\,(\ln|q|) \\ &\quad + \frac{p}{2\,q}\,H(\sigma,\zeta,\bar\zeta). \end{aligned} \tag{26}$$

Here $\alpha(\sigma)$, $\beta(\sigma)$, and $H(\sigma,\zeta,\bar\zeta)$ are arbitrary functions of the coordinates and λ is the cosmological constant. Additionally, we introduce the notion of the so-called propagation 1-form $k := k_\mu\vartheta^\mu$ which inherits the properties of the geodesic, shear-free, expansion-free and twistless null vector field k^μ representing the propagation vector of a plane-fronted wave. We proceed by imposing some restrictions on the electromagnetic 2-form F and a 2-form $S_{\alpha\beta}$ defined in terms of the irreducible decomposition of the Riemannian curvature 2-form $\tilde R_{\alpha\beta}$ in the following way:

$$\begin{aligned} S_{\alpha\beta} &:= \tilde R_{\alpha\beta} - {}^{(6)}\tilde R_{\alpha\beta} = {}^{(1)}\tilde R_{\alpha\beta} + {}^{(4)}\tilde R_{\alpha\beta} \\ &= \tilde R_{\alpha\beta} + \frac{1}{12}(e_\nu\rfloor e_\mu\rfloor \tilde R^{\nu\mu})\,\vartheta_\alpha\wedge\vartheta_\beta \overset{\text{in vacuum}}{=} {}^{(1)}\tilde R_{\alpha\beta} =: C_{\alpha\beta}. \end{aligned} \tag{27}$$

They shall obey the so-called radiation conditions

$$S_{\alpha\beta} \wedge k = 0, \quad \text{and} \quad (e_\alpha \rfloor k)\, S^{\alpha}{}_{\beta} = 0, \tag{28}$$

and

$$F \wedge k = 0, \quad \text{and} \quad \frac{1}{2}(e^{\alpha} \rfloor k)\, e_\alpha \rfloor F = 0. \tag{29}$$

If one imposes the conditions (28) and (29), the formerly arbitrary functions $\alpha(\sigma)$ and $\beta(\sigma)$ in (24)-(26) become restricted, namely $\alpha(\sigma)$ to the real and $\beta(\sigma)$ to the complex domain, see [5]. In the next step we insert the ansatz for the coframe (23) into the Einstein-Maxwell field equations with cosmological constant

$$\eta_{\alpha\beta\gamma} \wedge \tilde{R}^{\beta\gamma} + 2\,\lambda\,\eta_\alpha \;=\; 2\kappa\Sigma_\alpha^{\rm Max}, \tag{30}$$

$$dF = 0, \quad d\,{}^\star F \;=\; 0, \tag{31}$$

where $\Sigma_\alpha^{\rm Max}$ is the energy-momentum current of the electromagnetic field (cf. eq. (21)). Let us consider the vacuum field equations first before switching on the electromagnetic field, i. e. there are only gravitational waves and we only have to take into account the left hand side of (30). As shown in [5], this equation, after inserting the coframe into it (cf. eq. (4.34) of [5]), turns into a homogeneous PDE for the unknown function $H(\sigma,\zeta,\bar\zeta)$:

$$H_{,\zeta\bar\zeta} + \frac{\lambda}{3p^2} H = 0. \tag{32}$$

Equation (4.39) of [5] supplies us with the solution for $H(\sigma,\zeta,\bar\zeta)$ in terms of an arbitrary holomorphic function $\phi(\sigma,\zeta)$

$$H(\sigma,\zeta,\bar\zeta) = \phi_{,\zeta} - \frac{\lambda}{3}\frac{\bar\zeta}{p}\,\phi + \bar\phi_{,\bar\zeta} - \frac{\lambda}{3}\frac{\zeta}{p}\,\bar\phi. \tag{33}$$

Observe that H in (33) is a real quantity. We are now going to switch on the electromagnetic field. We make the following ansatz for the electromagnetic 2-form F in terms of an arbitrary complex function $f(\sigma,\zeta)$:

$$F = dA = -d\left[\left(\int^{\zeta} d\zeta'\, f(\zeta',\sigma) + \int^{\bar\zeta} d\bar\zeta'\, \bar f(\bar\zeta',\sigma)\right)\vartheta^{\hat 2}\right]. \tag{34}$$

In compliance with [5], this ansatz for F leads to $\Sigma_\alpha^{\rm Max} = -2\,\delta_\alpha^{\hat 2}\, p^2 f\bar f\,\vartheta^{\hat 0}\wedge\vartheta^{\hat 1}\wedge\vartheta^{\hat 2}$. Now the field equations (30)-(31) turn into an inhomogeneous PDE for $H(\sigma,\zeta,\bar\zeta)$ (cf. eq. (4.35) of [5]):

$$H_{,\zeta\bar\zeta} + \frac{\lambda}{3p^2} H = \frac{2\kappa p}{q}\, f\,\bar f. \tag{35}$$

The homogeneous solution $H_{\rm h}(\sigma,\zeta,\bar\zeta)$ of this equation is again given by (33). The particular solution $H_{\rm p}(\sigma,\zeta,\bar\zeta)$ of the inhomogeneous equation can be written in a similar form,

$$H_{\rm p}(\sigma,\zeta,\bar\zeta) = \mu_{,\zeta} - \frac{\lambda}{3}\frac{\bar\zeta}{p}\mu + \bar\mu_{,\bar\zeta} - \frac{\lambda}{3}\frac{\zeta}{p}\bar\mu, \tag{36}$$

where the function $\mu(\sigma,\zeta,\bar\zeta)$ can be expressed in the following integral form:

$$\mu(\sigma,\zeta,\bar\zeta)=\kappa\int^{\bar\zeta}d\bar\zeta p^2\int^{\zeta}d\zeta'\,\frac{1}{p^2}\int^{\zeta'}d\zeta''\,\frac{p\,f\,\bar f}{q}. \tag{37}$$

Of course, one is only able to derive $H_{\rm p}$ explicitly after choosing the arbitrary functions $\alpha(\sigma)$ and $\beta(\sigma)$ in (24)-(26). They enter the coframe and, as a consequence, the function $\mu(\sigma,\zeta,\bar\zeta)$ in (37). The general solution reads[1]

$$H(\sigma,\zeta,\bar\zeta)=H_{\rm h}(\sigma,\zeta,\bar\zeta)+H_{\rm p}(\sigma,\zeta,\bar\zeta). \tag{38}$$

We proceed with a particular choice for the functions entering the coframe and the ansatz for the electromagnetic potential

$$\alpha=1,\qquad \beta=0,\qquad f=f_0\,\zeta^n\quad\text{where}\quad \mathrm{n}=0,\pm1,\pm2,\dots, \tag{39}$$

and $[f_0]$ =length^{-2-n}. The electromagnetic potential is now given by

$$A=-f_0\left(\int^{\zeta}d\zeta'\,\zeta'^{\,n}+\int^{\bar\zeta}d\bar\zeta'\,\bar\zeta'^{\,n}\right)\vartheta^{\hat 2}. \tag{40}$$

The only unknown function is $H_{\rm p}(\sigma,\zeta,\bar\zeta)$. Thus we have to carry out the integration (cf. (7.7) of [5]) in eq. (37), after substituting the function $f=f_0\,\zeta^n$ which originates from our ansatz (39). The solution of this integration for different choices of n is given in (7.10)-(7.13) of [5]. We will present this solution in a more compact form as

(i) $n<-1$

$$\begin{aligned}H_{\rm p}&=\frac{2\kappa p f_0^2}{q}\left(\frac{(\zeta\bar\zeta)^{1+n}}{(1+n)^2}+4\left(\frac{\lambda}{6}\right)^{-n-1}\ln|\,q\,|-4\left(\frac{\lambda}{6}\right)^{-n-1}\ln|\,p-1\,|\right.\\&\quad\left.+4\sum_{r=1}^{-n-1}\frac{\left(\frac{\lambda}{6}\right)^{-n-r-1}}{r\,\left(\zeta\bar\zeta\right)^r}\right)+\frac{8\kappa f_0^2\,\left(\zeta\bar\zeta\right)^{n+1}}{(1+n)\,p},\end{aligned} \tag{41}$$

(ii) $n=-1$

$$H_{\rm p}=\frac{2\kappa f_0^2}{p}\left(4\,q\,\ln|\,q\,|+\frac{2\lambda\zeta\bar\zeta}{3}\ln\left(f_0^2\zeta\bar\zeta\right)+\frac{q}{2}\ln^2\left(f_0^2\zeta\bar\zeta\right)\right), \tag{42}$$

(iii) $n>-1$

$$\begin{aligned}H_{\rm p}&=\frac{8\kappa q f_0^2}{p}\left(\frac{\lambda}{6}\right)^{-n-1}\left(\ln|\,q\,|+\sum_{r=1}^{n}\frac{\binom{n}{r}}{r}\left((p-2)^r-(-1)^r\right)\right)\\&\quad+\frac{2\kappa f_0^2(\zeta\bar\zeta)^{n+1}}{p\,(n+1)^2}(4(n+1)+q).\end{aligned} \tag{43}$$

[1]We changed the name of the vacuum solution from $H(\sigma,\zeta,\bar\zeta)$, mentioned in (33), into $H_{\rm h}(\sigma,\zeta,\bar\zeta)$.

Note that the original solution in [5] is not completely correct, as admitted by Ozsváth [6]. Needless to say that we displayed in (41)-(43) the correct expressions for $H_{\rm p}$. We characterize the solutions obtained above by means of the selfdual part of the conformal curvature 2-form ${}^{+}C_{\alpha\beta}$, the trace-free Ricci 1-form $\tilde{R\!\!\!/}_{\alpha}$, and the Ricci scalar $\tilde{R}$. Here we list only the results for the general ansatz (24)-(26), i. e. for arbitrary $\alpha(\sigma)$, $\beta(\sigma)$, and $H(\sigma,\zeta,\bar{\zeta})$. For the sake of brevity we make use of the structure functions p and q as defined in (24) and (25):

$$
\begin{aligned}
{}^{+}C_{\hat{2}\hat{0}} &= -{}^{+}C_{\hat{0}\hat{2}} = \frac{1-i}{4}\,\partial_{\zeta}\left[q^{2}\partial_{\zeta}\left(\frac{p}{q}H\right)\right]\,\vartheta^{\hat{0}}\wedge\vartheta^{\hat{2}}, && (44)\\
{}^{+}C_{\hat{2}\hat{1}} &= -{}^{+}C_{\hat{1}\hat{2}} = \frac{1+i}{4}\,\partial_{\bar{\zeta}}\left[q^{2}\partial_{\bar{\zeta}}\left(\frac{p}{q}H\right)\right]\,\vartheta^{\hat{1}}\wedge\vartheta^{\hat{2}}, && (45)\\
\tilde{R\!\!\!/}_{\hat{2}} &= \frac{2pq\kappa p}{q}\,f\,\bar{f}\,\vartheta^{\hat{2}} = 2\kappa p^{2}\,f\,\bar{f}\,\vartheta^{\hat{2}}. && (46)
\end{aligned}
$$

5 Plane-fronted waves in MAG

We now turn to the triplet subcase of MAG. Thus we are concerned with the triplet of 1-forms in eq. (14). As shown in section 3, the triplet ansatz reduces the electrovacuum MAG field equations (4) and (5) to an effective Einstein-Proca-Maxwell system:

$$
\begin{aligned}
a_0\,\eta_{\alpha\beta\gamma}\wedge\tilde{R}^{\beta\gamma}+2\lambda\eta_{\alpha} &= 2\kappa\left[\Sigma_{\alpha}^{(\omega)}+\Sigma_{\alpha}^{\rm Max}\right], && (47)\\
d\,{}^{\star}d\omega+m^{2}\,{}^{\star}\omega &= 0, && (48)\\
dF=0,\quad d\,{}^{\star}F &= 0. && (49)
\end{aligned}
$$

From here on we will presuppose that $m^2=0$, reducing eq. (48) to $d\,{}^{\star}d\omega=0$ and the energy-momentum $\Sigma_{\alpha}^{(\omega)}$ of the triplet field to the first line of eq. (20). As one realizes immediately, the system (47)-(49) now becomes very similar to the one investigated in the Einstein-Maxwell case in (30)-(31). Let us start with the same ansatz for the line element, coframe (22)-(26), and electromagnetic 2-form (34). The only thing missing up to now is a suitable ansatz for the 1-form ω which governs the non-Riemannian parts of the system and enters eqs. (47)-(48):

$$
\omega=-\left[\int^{\zeta}d\zeta^{'}\,g(\sigma,\zeta^{'})+\int^{\bar{\zeta}}d\bar{\zeta}^{'}\,\bar{g}(\sigma,\bar{\zeta}^{'})\right]\vartheta^{\hat{2}}. \qquad (50)
$$

Here $g(\sigma,\zeta)$ represents an arbitrary complex function of the coordinates. Since the first field equation (47) in the MAG case differs from the Einsteinian one only by the emergence of $\Sigma_{\alpha}^{(\omega)}$. Accordingly, we expect only a linear change in the PDE (35). Thus, in case of switching on the electromagnetic and the triplet field, the solution for $\mathcal{H}(\sigma,\zeta,\bar{\zeta})$ entering the coframe, is determined by

$$
\mathcal{H}_{,\zeta\bar{\zeta}}+\frac{\lambda}{3p^{2}}\mathcal{H}=\frac{2\kappa p}{q}\left(f\bar{f}+g\bar{g}\right). \qquad (51)
$$

We use calligraphic letters for quantities that belong to the MAG solution. Consequently, the homogeneous solution $\mathcal{H}_{\rm h}$, corresponding to $f = g = 0$, is given again by (33). In order to solve the inhomogeneous equation (51), we modify the ansatz for $H_{\rm p}$ made in (36) and (37). For clarity, we distinguish between the Einstein-Maxwell and the MAG case by changing the name of $\mu(\sigma,\zeta,\bar\zeta)$ in (36) into $M(\sigma,\zeta,\bar\zeta)$ which leads to the following form of $\mathcal{H}_{\rm p}$:

$$\mathcal{H}_{\rm p}(\sigma,\zeta,\bar\zeta) = M_{,\zeta} - \frac{\lambda}{3}\frac{\bar\zeta}{p}M + \bar M_{,\bar\zeta} - \frac{\lambda}{3}\frac{\zeta}{p}\bar M, \tag{52}$$

$$\text{where}\quad M = \kappa\int^{\bar\zeta} d\bar\zeta p^2 \int^{\zeta} d\zeta' \frac{1}{p^2}\int^{\zeta'} d\zeta''\frac{p}{q}\left(f\bar f + g\bar g\right). \tag{53}$$

Thus, the general solution of (51) is given by

$$\mathcal{H}(\sigma,\zeta,\bar\zeta) = \mathcal{H}_{\rm h}(\sigma,\zeta,\bar\zeta) + \mathcal{H}_{\rm p}(\sigma,\zeta,\bar\zeta). \tag{54}$$

Substitution of this ansatz into the field equations yields the following constraint for the coupling constants of the constrained MAG Lagrangian:

$$a_0 = 1, \quad z_4 = \frac{\rho}{2k_0}. \tag{55}$$

Here, we made use of the definition of k_0 mentioned in (19). We will now look for a particular solution $\mathcal{H}_{\rm p}$ of (51). As in the Riemannian case, we choose $\alpha = 1$, $\beta = 0$ and make a polynomial ansatz for the functions f and g which govern the Maxwell and triplet regime of the system,

$$f = f_0\zeta^n \quad n = 0,\pm1,\pm2,\ldots, \quad g = g_0\zeta^l \quad l = 0,\pm1,\pm2,\ldots\;, \tag{56}$$

with $[g_0]$ =length^{-2-l}. Now we have to perform the integration in (53) which yields the solution for $\mathcal{H}_{\rm p}$ via eq. (52). At this point we remember that the solution for $\mathcal{H}_{\rm p}$, in case of excitations corresponding to f, is already known from eqs. (41)-(43). Let us introduce a new name for $H_{\rm p}$ as displayed in (41)-(43), namely $\mathcal{H}^f_{\rm p}$. Furthermore, we introduce the quantity $\mathcal{H}^g_{\rm p}$, defined in the same way as $\mathcal{H}^f_{\rm p}$ but with the ansatz for g from (56). Thus, one has to perform the substitutions $n \to l$ and $f_0 \to g_0$ in eqs. (41)-(43) in order to obtain $\mathcal{H}^g_{\rm p}$. Due to the linearity of our ansatz in (53), we can infer that the particular solution $\mathcal{H}_{\rm p}$ in the MAG case is given by the sum of the appropriate branches for $\mathcal{H}^f_{\rm p}$ and $\mathcal{H}^g_{\rm p}$, i.e. the general solution form eq. (54) now reads

$$\mathcal{H}(\sigma,\zeta,\bar\zeta) = \mathcal{H}_{\rm h}(\sigma,\zeta,\bar\zeta) + \mathcal{H}^f_{\rm p}(\sigma,\zeta,\bar\zeta) + \mathcal{H}^g_{\rm p}(\sigma,\zeta,\bar\zeta). \tag{57}$$

Note that we have to impose the same additional constraints among the coupling constants as in the general case (cf. eq. (55)). In contrast to the general relativistic case, there are two new geometric quantities entering our description, namely the torsion T_α and the nonmetricity $Q_{\alpha\beta}$ given by

$$Q_{\alpha\beta} \;=\; -\frac{4k_1}{9}\vartheta_{(\alpha}e_{\beta)}\rfloor\left[\int^{\zeta} d\zeta' g(\sigma,\zeta') + \int^{\bar\zeta} d\bar\zeta' \bar g(\sigma,\bar\zeta')\right]\vartheta^{\hat 2}$$

$$
+g_{\alpha\beta}\left(\frac{k_1}{9}-k_0\right)\left[\int^{\zeta} d\zeta^{'}\, g(\sigma,\zeta^{'})+\int^{\bar{\zeta}} d\bar{\zeta}^{'}\,\bar{g}(\sigma,\bar{\zeta}^{'})\right]\vartheta^{\hat{2}}, \quad (58)
$$

$$
T^{\alpha} = -\frac{k_2}{3}\left[\int^{\zeta} d\zeta^{'}\, g(\sigma,\zeta^{'})+\int^{\bar{\zeta}} d\bar{\zeta}^{'}\,\bar{g}(\sigma,\bar{\zeta}^{'})\right]\vartheta^{\alpha}\wedge\vartheta^{\hat{2}}. \quad (59)
$$

6 Summary

We investigated plane-fronted electrovacuum waves in MAG with cosmological constant in the triplet ansatz sector of the theory. The spacetime under consideration carries curvature, nonmetricity, torsion, and an electromagnetic field. Apart from the cosmological constant, the solution contains several arbitrary functions, namely $\alpha(\sigma)$, $\beta(\sigma)$, $f(\sigma,\zeta)$, $g(\sigma,\zeta)$, and $\phi(\sigma,\zeta)$. One may address these functions by the generic term *wave parameters* since they control the different sectors of the solution like the electromagnetic or the non-Riemannian regime. In this way, we generalized the class of solutions obtained by Ozsváth, Robinson, and Rózga [5].

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