Lecture
General Relativity – Field equations

Dirk Puetzfeld (ZARM, U Bremen)

GEO-Q Lecture Week, June 2016

Comments and corrections regarding this handout are welcome!

Contact information:

dirk.puetzfeld@zarm.uni-bremen.de
http://puetzfeld.org

Geometry and tensors

- Coordinates

$\bullet\, x^a \qquad \bullet\, x^b \qquad \bullet\, x^c$

Set of $a, b, c = 1, \ldots, n$ numbers

- Different values, depending on CS
- We want to formulate physical laws in a form invariant manner
- We want formalism which applies to any CS (not only Cartesian)

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Geometry and tensors

- Scalar

$\bullet\, \varphi(x^a) \qquad \bullet\, \varphi(x^b) \qquad \bullet\, \varphi(x^c)$

Single numerical value, independent of coordinate system

$$\varphi = \tilde{\varphi}$$

Example: Charge of a particle

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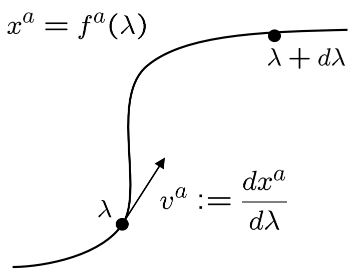
Geometry and tensors

- Contravariant vector



v^a tangent vector to the curve is an prototype of what one contravariant vector

Example: Velocity of a particle

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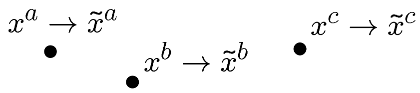
Geometry and tensors



$\tilde{x}^a = f^a(x^b)$ $\quad a, b = 1, \ldots, n$ coordinate transformation

Def: Contravariant vector v^a (quantity with n components)

$$\tilde{v}^a = \sum_b \frac{\partial \tilde{x}^a}{\partial x^b} v^b$$

which components in different CS are related by this transformation law for any coordinate transformation

Geometry and tensors

Def: w_a covariant vector

$$\sum_a w_a v^a = \sum_a \tilde{w}_a \tilde{v}^a$$

for all contravariant vectors v^a and all $x^a \to \tilde{x}^a$

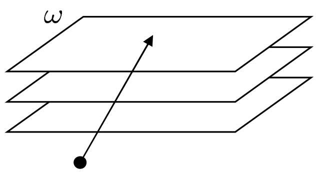
Remark: The sum is called the scalar product of v^a and w_a

→ transformation behavior

$$\tilde{w}_a = \sum_b \frac{\partial x^b}{\partial \tilde{x}^a} w_b$$

Geometry and tensors

- **Covariant vector**



w_a covariant vector has the transformation behavior of a 1-form $\omega = w_a dx^a$

Example: Momentum of a particle

Geometry and tensors

Def: δ^a_b Kronecker (1823-1891) symbol

$$\delta^a_b := \begin{cases} 1 & a = b \\ 0 & \text{else} \end{cases}$$

Convention: Summation over repeated indices

$$X^{\cdots\cdots}{}_{\cdots a \cdots} Y^{\cdots a \cdots}{}_{\cdots\cdots} = \sum_a X^{\cdots\cdots}{}_{\cdots a \cdots} Y^{\cdots a \cdots}{}_{\cdots\cdots}$$

Geometry and tensors

Now we are ready for objects with more than one index (multilinear objects)

Def: $T^{a_1 \cdots a_k}$ contravariant tensor of rank k

$$T^{a_1 \cdots a_k} C^{(1)}_{a_1} \cdots C^{(k)}_{a_k} = \tilde{T}^{a_1 \cdots a_k} \tilde{C}^{(1)}_{a_1} \cdots \tilde{C}^{(k)}_{a_k}$$

if the sum with k covariant $C^{(i)}_{a_{(i)}}$ vectors is a scalar

- The word "tensor" was introduced by W.R. Hamilton in the context of algebraic systems in 1846
- In its current usage by W. Voigt in 1899
- Tensor calculus developed ~ 1890 by G. Ricci-Cubastro (1853-1925)

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Geometry and tensors

Example: Transformation behavior

$$\tilde{T}^{ab} = \frac{\partial \tilde{x}^a}{\partial x^c} \frac{\partial \tilde{x}^b}{\partial x^d} T^{cd}$$

Example: Electromagnetic field

$$F^{ab} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^x & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}$$

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Geometry and tensors

Def: $T^{a_1 \cdots a_k}{}_{a_{k+1} \cdots a_{k+m}}$ mixed tensor of rank (k, m)

$$T^{a_1 \cdots a_k}{}_{a_{k+1} \cdots a_{k+m}} C^{(1)}_{a_1} \cdots C^{(k)}_{a_k} C^{a_{k+1}}_{(k+1)} \cdots C^{a_{k+m}}_{(k+m)} = \tilde{T}^{a_1 \cdots a_k}{}_{a_{k+1} \cdots a_{k+m}} \tilde{C}^{(1)}_{a_1} \cdots \tilde{C}^{(k)}_{a_k} \tilde{C}^{a_{k+1}}_{(k+1)} \cdots \tilde{C}^{a_{k+m}}_{(k+m)}$$

if the sum with k contravariant $C^{(i)}_{a_{(i)}}$ vectors and m covariant vectors $C^{a_{(i)}}_{(i)}$ is a scalar

Example: Transformation behavior rank $(1,1)$

$$\tilde{T}^a{}_b = \frac{\partial \tilde{x}^a}{\partial x^c} \frac{\partial x^d}{\partial \tilde{x}^b} T^c{}_d$$

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Geometry and tensors

- Important: So-far introduced all notions in general **affine** (not only for metric) **spaces**

- Rules:
  1. Product of two tensor

     $$(a,b) \times (c,d) \longrightarrow (a+c, b+d)$$

  2. Contraction of a tensor

     $$T^{\cdots a \cdots}{}_{\cdots a \cdots} \qquad (a,b) \longrightarrow (a-1, b-1)$$

  3. Sum of two tensors

     $$(a,b) \longrightarrow (a,b)$$

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Geometry and tensors

Def: Antisymmetric part (of a rank 2 tensor)

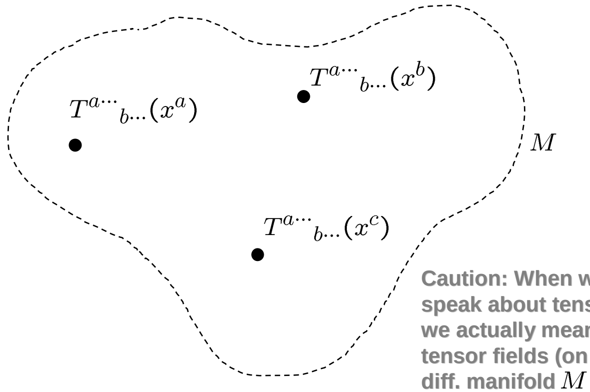
$$T_{[ab]} := \frac{1}{2}\left(T_{ab} - T_{ba}\right)$$

Symmetric part (of a rank 2 tensor)

$$T_{(ab)} := \frac{1}{2}\left(T_{ab} + T_{ba}\right)$$

- The (anti-)symmetry property will be conserved in all CS
- If $T_{[ab]} = T_{ab}$ we call T_{ab} totally antisymmetric (# components $\frac{n(n-1)}{2}$)
- If $T_{(ab)} = T_{ab}$ we call T_{ab} totally symmetric (# components $\frac{n(n+1)}{2}$)
- Example: Electromagnetic field $F^{ab} = F^{[ab]}$

Geometry and tensors



Caution: When we speak about tensors we actually mean tensor fields (on some diff. manifold M)

Geometry and tensors

Question: Is it possible to construct new tensor fields by differentiating given ones?

Example 1: Scalar field $\phi\left(x^a\right)$ $(0,0)$

$$\frac{\partial \phi}{\partial x^a} := \phi_{,a}$$

Answer: Yes! Yields a covariant vector field $(0,1)$

(Because of $d\phi = \phi_{,a} dx^a$)

Geometry and tensors

Example 2: Covariant vectorfield $v_a\left(x^b\right)$ $(0,1)$

$$v_{a,b} = \frac{\partial v_a}{\partial x^b} \qquad x^a \to \tilde{x}^b$$

Answer: No! Does not yield a covariant tensor field $(0,2)$

$$\tilde{v}_{a,b} = \frac{\partial^2 x^c}{\partial \tilde{x}^a \partial \tilde{x}^b} v_c + \frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial x^d}{\partial \tilde{x}^b} v_{c,d}$$

→ Since we are not only interested in linear but general coordinate transformations we need something new

Geometry and tensors

$$\tilde{v}_{a,b} - \underbrace{\frac{\partial^2 x^c}{\partial \tilde{x}^a \partial \tilde{x}^b}\frac{\partial \tilde{x}^e}{\partial x^c}}_{=:\, \tilde{\gamma}^e_{ab}} \tilde{v}_e = \frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial x^d}{\partial \tilde{x}^b} v_{c,d}$$

Assume: $\exists \quad (0,2)-\text{tensor} \quad \text{in} \quad x^a : A_{ab} = v_{a,b}$

$$\longrightarrow \quad \tilde{A}_{ab} = \tilde{v}_{a,b} - \tilde{\gamma}^c_{ab}\tilde{v}_c$$

This clearly shows that in CS $\tilde{x}^a$ we have to introduce a new coordinate dependent quantity

Geometry and tensors

Strategy: Introduce non-tensorial quantity Γ^c_{ab} to fix things in all CS

Requirement: $A_{ab} = v_{a,b} - \Gamma^c_{ab} v_c$ tensor in CS x^a

$\tilde{A}_{ab} = \tilde{v}_{a,b} - \tilde{\Gamma}^c_{ab} \tilde{v}_c$ tensor in CS $\tilde{x}^a$

$\downarrow$ Derive transformation properties of Γ^c_{ab}

$$\tilde{\Gamma}^c_{ab} = \frac{\partial x^d}{\partial \tilde{x}^a}\frac{\partial x^e}{\partial \tilde{x}^b}\frac{\partial \tilde{x}^c}{\partial x^f}\Gamma^f_{de} + \frac{\partial \tilde{x}^c}{\partial x^f}\frac{\partial^2 x^f}{\partial \tilde{x}^a \partial \tilde{x}^b}$$

Affine connection

Geometry and tensors

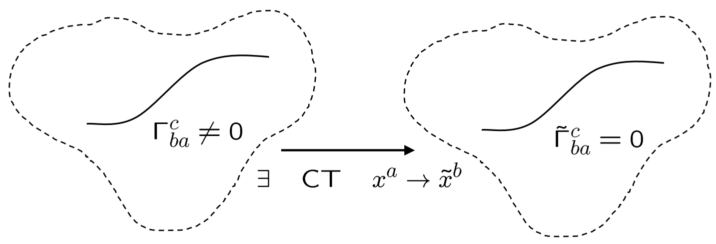
- Properties of the affine connection:

1. n^3 independent components
2. Difference of two connections is a tensor $_1\Gamma^c_{ab} - {_2\Gamma^c_{ab}}$
3. Γ^c_{ab} connection then Γ^c_{ba} connection
4. $\Gamma^c_{(ab)}$ is a symmetric connection ($\frac{n^2(n+1)}{2}$ components)
5. $\Gamma^c_{[ab]}$ is a tensor (so-called torsion)
6. A general non-symmetric connection can be written in the form

$$\Gamma^c_{ab} = \Gamma^c_{(ab)} + \Gamma^c_{[ab]}$$

Geometry and tensors

- Theorem



Even for non-symmetric connections!

Geometry and tensors

Def: The tensor of rank $(0,2)$

$$\nabla_b v_a = v_{a;b} := v_{a,b} - \Gamma^c_{ab} v_c$$

is called <u>covariant derivative</u> of the VF v_a

- Properties:

1. $\phi_{;a} = \phi_{,a}$ for scalars
2. For tensors $(A^{\cdots}_{\cdots} B^{\cdots}_{\cdots})_{;a} = A^{\cdots}_{\cdots;a} B^{\cdots}_{\cdots} + A^{\cdots}_{\cdots} B^{\cdots}_{\cdots;a}$

→ Derive rules for contravariant VF and mixed tensors

Geometry and tensors

Def: Covariant derivative of a tensor of rank (p,q)

$$\nabla_k T^{ab\cdots}{}_{cd\ldots} = T^{ab\cdots}{}_{cd\ldots,k} + \Gamma^a_{lk} T^{lb\cdots}{}_{cd\ldots} + \ldots - \Gamma^l_{ck} T^{ab\cdots}{}_{ld\ldots} - \ldots$$

- Warning: In general covariant derivates of an object <u>do not</u> commute!

$$T^{\cdots}_{\cdots;ab} \neq T^{\cdots}_{\cdots;ba}$$

- Covariant derivative increases rank $(a,b) \stackrel{\nabla}{\longrightarrow} (a,b+1)$

Geometry and tensors

Def: Covariant derivative of a tensor of rank (p,q)

$$\nabla_k T^{ab\cdots}{}_{cd\ldots} = T^{ab\cdots}{}_{cd\ldots,k} + \Gamma^a_{lk} T^{lb\cdots}{}_{cd\ldots} + \ldots - \Gamma^l_{ck} T^{ab\cdots}{}_{ld\ldots} - \ldots$$

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- Covariant derivative increases rank $(a,b) \stackrel{\nabla}{\longrightarrow} (a,b+1)$

Geometry and tensors

- Connection plays role in the parallel transport of quantities (infinitesimally):

$$\delta v^a = -\Gamma^a_{bc} v^b dx^c$$

Def: Absolute derivative & parallel transport

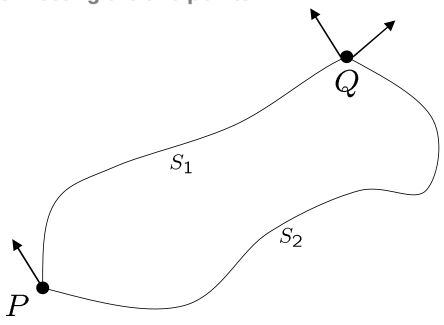
$$\frac{\delta}{\delta s} T^a := \frac{dx^b}{ds} \nabla_b T^a = 0$$

$:\Leftrightarrow T^a$ is parallelly transported along curve $x^a(s)$

$$(a,b) \stackrel{\frac{\delta}{\delta s}}{\longrightarrow} (a,b)$$

Geometry and tensors

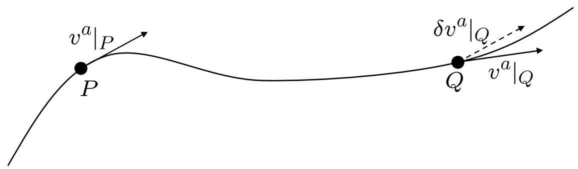
- **Careful:** In general the parallel transport will <u>not only</u> depend on the endpoints P and Q but also on the curve connecting the two points



Geometry and tensors

- Now consider a special type of curves, namely those along which the tangent vector to the curve equals the parallelly transported tangent vector at any point of the curve

Def: Autoparallel is a curve $:\Leftrightarrow v^a|_Q = \delta v^a|_P$



Think of those curves as the "straightest" possible curves

Geometry and tensors

- Differential equation which has to be fulfilled at every point of these special curves:

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = g(s)\frac{dx^a}{ds}$$

- Remarks:

1. Its solution will be completely determined by a point P and the direction of the tangent vector at P
2. If Γ^a_{bc} is a non-symmetric connection only the symmetric part contributes to this equation

- Simpler form by reparametrization $s = s(\sigma)$, choose $\frac{d^2\sigma}{ds^2} = g(s)\frac{d\sigma}{ds}$

$$\frac{d^2x^a}{d\sigma^2} + \Gamma^a_{bc}\frac{dx^b}{d\sigma}\frac{dx^c}{d\sigma} = 0$$

Geometry and tensors

- **Task:** Compute anti-symmetric part of the 2[nd] covariant derivative of a VF

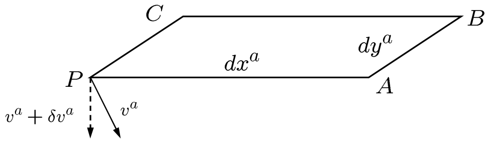
$$\underset{(0,3)}{2v_{a;[bc]}} = \underbrace{-2\Gamma^d_{a[b,c]}v_d - 2\Gamma^d_{a[b}\Gamma^e_{d|c]}v_e}_{=:\, R^d{}_{abc}\ \ v_d} - \underset{(1,2)}{2\Gamma^d_{[bc]}}\underset{(0,2)}{v_{a;d}}$$

$R^d{}_{abc} \rightarrow (1,3)$, v_d: $(0,1)$

- Remarks

1. Antisymmetric in last two indices $R^d{}_{abc} = -R^d{}_{acb}$
2. If $\Gamma^c_{ab} = \Gamma^c_{(ab)}$ then $R^d{}_{[abc]} = 0$
3. Two possible contractions $R^a{}_{abc}$ $R^a{}_{bca}$

Geometry and tensors

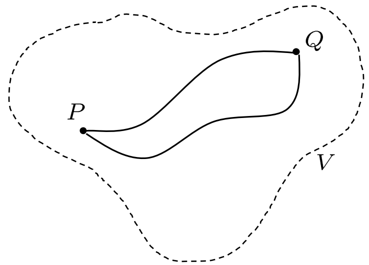


$$\delta v^a = -\frac{1}{2} R^a{}_{bcd} v^b \left(dx^c dy^d - dx^d dy^c \right)$$

Geometry and tensors

- **Theorem 1:**

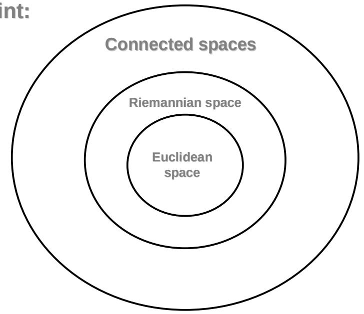
  The result of <u>parallel transport</u> between P and Q along paths which lie entirely in V is <u>path independent</u> if $R^d{}_{abc} = 0$ in V (necessary and sufficient)



- **Theorem 2:** $\Gamma^c_{ab} = \Gamma^c_{(ab)} \quad \wedge \quad R^d{}_{abc} = 0 \quad \text{in V} \quad \Rightarrow \exists x^a \to \tilde{x}^a : \Gamma^c_{ab} = 0$

Geometry and tensors

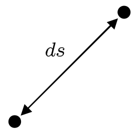
- **Up to this point:**



- **Metric space: Space in which a prescription is given to attribute a scalar distance to each pair of neighboring points**

Geometry and tensors

- **Examples:**



1. **3d Euclidean space (in Cartesian CS)**

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

2. **4d Minkowski space (in Cartesian CS)**

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

In general coordinates: $ds^2 = g_{ab} dx^a dx^b$ **line element**

Geometry and tensors

Metric tensor	g_{ab}	$(0,2)$	$\frac{n(n+1)}{2}$

- In order to uniquely determined it needs to be symmetric
- In general Riemannian spaces the metric can be arbitrarily functions of the coordinates
- In general not possible to reduce them to the simple form as in Minkowski or Euclidean space
- Allows us to construct a scalar $g_{ab}\,{}_1dx^a\,{}_2dx^b$, which generalizes the scalar product from Euclidean space
- Remember: In affine spaces (non-metric!) we could form the scalar product only from a covariant and a contravariant vector
- Allows for the definition of a covariant equivalent to a contravariant vector:

$$g_{ab}v^b = v_a$$

Geometry and tensors

- Hence we can raise and lower indices with the metric, generalization to tensors of any rank is straightforward:

$$T^{ab\cdots}{}_{ef\cdots}\, g_{bk} = T^{a}{}_{k}{}^{\cdots}{}_{ef\cdots}$$

- Note: The fundamental distinction between contravariant and covariant tensors does not exist in Riemannian spaces!
- In analogy to Euclidean space we can postulate that the scalar product defines the angle between two vectors

$$\cos(X,Y) = \frac{g_{ab}X^aY^b}{\sqrt{|g_{ab}X^aX^b|}\sqrt{|g_{ab}Y^aY^b|}}$$

- The contravariant form of the metric is given by:

$$g_{ab}g^{bc} = \delta_a^c$$

- We will actually work in pseudo-Riemannian spacetime, i.e. metric can be (positive, negative, in)-definite

Geometry and tensors

- For a given metric, and at any given point P, it is always possible to find a CT such that the transformed metric has diagonal form with +1 and -1 on the diagonal

$$\exists \quad \text{CT} \quad x^a \to \tilde{x}^b : \qquad \tilde{g}_{ab}|_P = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- In the diagonal form the number of +1 and -1 does not change if we only consider real transformations
- Difference of the #minus and #plus is called signature of spacetime (examples: (i) Euclidean space: 0 (ii) Minkowski space: 2)
- Recall: In general parallel transport defined w.r.t. to a general connection. In Riemannian space there is a special connection derived directly from the metric tensor:

Christoffel-symbols	$\Gamma_{ab}{}^{c} = \frac{1}{2} g^{cd}\left(g_{db,a} + g_{ad,b} - g_{ab,d}\right)$

Geometry and tensors

- This connection is used exclusively in Riemannian geometry
- The Riemannian connection is symmetric, and makes the metric covariantly constant:

$$\nabla_a g_{bc} = 0$$

- Of course one can make the Riemannian connection vanish in special CS (the more general statement regarding affine connections was already discussed)
- We are now able to measure lengths in Riemannian space

Def: Proper length	$s := \int_P^Q ds$

- We can use the proper length as a curve parameter, the tangent vector along a curve is normalized if we use the proper length as a curve parameter

Geometry and tensors

- Now: Use the proper length in the autoparallel equation
- Result: The proper length s is an affine parameter of the autoparallel

$$\frac{du^a}{ds} + \Gamma_{bc}{}^a u^b u^c = 0 \quad \Leftrightarrow \quad u^a{}_{;b} u^b = 0$$

Geodesic equation

$$u^b =: \frac{dx^b}{ds}$$

- In Riemannian spaces the connection will be given by the Christoffel symbol
- Autoparallels in Riemannian space will be called geodesics
- Geodesics between two points are curves of extremal length

Geometry and tensors

- Curvature in a Riemannian space

$$R^a{}_{bcd} = R^a{}_{bcd}(\Gamma, \partial\Gamma) \quad \rightarrow \quad R^a{}_{bcd}\left(g, \partial g, \partial^2 g\right)$$

- Properties

1. $R^d{}_{abc} = -R^d{}_{acb}$
2. $R^d{}_{[abc]} = 0$
3. $R_{abcd} = g_{ad} R^d{}_{bca}$
4. $R_{abcd} = -R_{bacd} = R_{cdab}$
5. $R^a{}_{abc} = 0$
6. $R^a{}_{b[cd;e]} = 0$

Geometry and tensors

- Other tensors build from the curvature tensor

$$R_{ab} := R^a{}_{bca}$$

Ricci tensor

- Properties

1. Only non-vanishing contraction!
2. $R_{ab} = R_{ba}$

$$R := R^a{}_a$$

Ricci scalar

Geometry and tensors

- Other tensors build from the curvature tensor

$$G_{ab} := R_{ab} - \frac{1}{2} g_{ab} R$$

Einstein tensor

- Properties

1. $G_{ab} = G_{ba}$
2. $G^{ab}{}_{;b} = 0$

Geometry and tensors

- Other tensors build from the curvature tensor

$$C_{abcd} = R_{abcd} + \frac{2}{n-2}\left(g_{d[a}R_{b]c} - g_{c[a}R_{b]d}\right)\frac{2}{(n-1)(n-2)}g_{c[a}g_{b]d}R$$ Weyl tensor

- Properties

1. $C_{abcd} = -C_{bacd} = C_{cdab}$
2. $C_{a[bcd]} = 0$
3. $\longrightarrow$ $C^{a}{}_{bca} = 0$
4. For $\tilde{g}_{ab} = \varphi^2 g_{ab}:\quad \tilde{C}^{a}{}_{bcd} = C^{a}{}_{bcd}$

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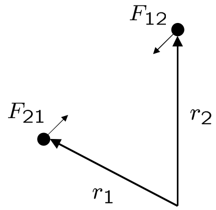
Newton‘s theory

- Theory formulated in the form of a force law



$$F_{21} = G\frac{m_1 m_2}{|r|^2}\frac{r}{|r|} = -F_{12}$$

$$r = r_2 - r_1$$

Coupling constant

$$G \approx 6.673 \times 10^{-11}\frac{m^2}{kg\,s^2}$$

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Newton‘s theory

- Now: Introduce gravitational field, i.e. force per unit mass

$$m_1 = m \qquad m_2 = M$$

$$f := \frac{F}{m} = \frac{GM}{|r|^2}\frac{r}{|r|}$$ “quasi” field theoretical notion

$$F_{mM} = mf$$

f may be expressed by a potential

$$\phi = -G\frac{M}{|r|} \quad \rightarrow \quad f = -\nabla\phi$$

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Newton‘s theory

Assume that the mass at r generates the gravitational field

$$\rightarrow \quad \nabla f = 4\pi G M \delta^3(r)$$ for a point mass

$$\rightarrow \quad \nabla^2\phi = -4\pi G\rho(r)$$ for a continuous matter distribution

- Properties:
  1. Scalar theory, since we have scalar potential and therefore a scalar source of the field
  2. FEQ is a linear partial differential equation of 2nd order
  3. Theory based on pre-relativistic concepts of absolute space and time:
     1. Field ϕ has no dynamical properties
     2. Newton’s theory represents an “action-at-a-distance” theory

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Road to GR

- Keep in mind: Newton's theory is an extremely successful theory (in particular when it comes to the description of the motion of planets)

- **Task:** Formulate a generalization of Newton's theory which makes use, and goes beyond, the successful concept of spacetime as encountered in SR

- Make sure:

  That the new theory reduces to Newton's theory in some (well-defined) limit

Road to GR

- Guideline: Special Relativity

  In SR continuous matter is described by a stress-energy-momentum tensor

$$ {}^{\text{Max}}T_{ab} = \begin{pmatrix} T_{00} & T_{0\alpha} \\ T_{\alpha 0} & T_{\alpha\beta} \end{pmatrix} $$

Energy density | Momentum density

Energy flux density | Momentum flux density = stresses

$$ {}^{\text{Max}}T_{ab} = F_{ac}F_b{}^c + \frac{1}{4} g_{ab} F_{cd} F^{cd} $$

→ Now: Use a similar structure in the construction of a relativistic theory of gravitation

Road to GR

- Attempt 1: Scalar theory (on a flat background)

$$ T = \eta^{ab} T_{ab} = T^a{}_a $$

↑ T: scalar source, close to the situation in Newton's theory

↑ η^{ab}: flat Minkowski metric as in SR

FEQ:

$$ \Box\phi = \partial_a \left(\eta^{ab} \partial_b \phi \right) = \kappa T $$

↑ ϕ: single gravitational potential

↑ κ: coupling constant

(Historical remark: Such a theory was first proposed by Nordström)

Road to GR

- Properties:

  \+ truly relativistic theory

  \+ reduces to Newtonian theory in a first approximation

$$ \Box\phi = \partial_a \left(\eta^{ab} \partial_b \phi \right) = \underbrace{\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}}_{= 0 \quad c \to \infty} - \nabla^2 \phi $$

  \- is in disagreement with observations!

    - predicts wrong perihelion precession of mercury: wrong sign as and wrong value
    - zero value for the deflection of light

Road to GR

- Attempt 2: Vector theory

There is <u>no straightforward</u> way to construct a vector from the EM tensor

↓

We would have to introduce some additional quantities over which we contract one of the indices

(Note: Nowadays there are several theories with vectorial aspects in gravity)

Road to GR

- Attempt 3: Tensorial theory

Rather natural to consider such a step since the previous attempts (scalar / vector) did not work

10 components of T_{ab} (which act as source of the gravitational field) → Corresponding gravitational potential should also be a symmetric tensor of rank $(0,2)$

→ The metric g_{ab} could play this role!

TASK: Formulate FEQ of this theory → This will be the FEQ of GR

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Note: Historically the so-called <u>E</u>quivalence <u>P</u>rinciple (EP) played an important role in the development of GR

The equivalence principle can be viewed as a generalization of an assumption which was already very successful in the Newtonian theory

gravitational mass	=	inertial mass
$F = G\dfrac{mM}{\lvert r\rvert^2}\dfrac{r}{\lvert r\rvert}$		$F = ma$

Assumption in Newton's theory

- Indirect verification: Success of Newton's theory, e.g. description of planetary motions
- Direct verification: $\dfrac{m_{\rm gr} - m_{\rm in}}{m_{\rm gr}}$

Eötvös (1922) $< 10^{-8}$

Dicke (1964) $< 10^{-11}$

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Equivalence Principle: Gravitational and inertial forces are completely equivalent, i.e. they are of an identical nature and consequently it is impossible to separate them in an physical experiment.

Einstein's generalization of the Newtonian equivalence principle

- Consequences:

1. Gravitational forces (or equivalently gravitational accelerations) should be described in the same way as inertial forces (or inertial accelerations)

$$\frac{d^2x^a}{ds^2} = 0$$

(free particle EOM in an inertial frame in Minkowski space)

↓ introduce general coordinates

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Road to GR

↓ introduce general coordinates

$$\frac{d^2x^a}{ds^2} + \Gamma_{bc}{}^{a}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0$$

(EOM in general coordinates, i.e. geodesic equation)

inertial acceleration of the particle, due to the fact that we use a non-inertial frame

→ $\Gamma_{bc}{}^{a}$ linked to inertial acceleration

Switch on gravity ↓

$\Gamma_{bc}{}^{a}$ →(EP) also describe gravitational accelerations

↓

Since the Christoffel symbols can be derived from the metric tensor, we conclude that the metric g_{ab} plays the role of the gravitational potential

Remark: The Christoffel symbols will describe the sum of inertial and gravitational accelerations. According to the EP it will not be possible to split this sum unambiguously by any experiment into the inertial and gravitational terms.

Road to GR

- Consequences:

2. When there are gravitational accelerations present, space cannot be the flat Minkowski spacetime

Gravitational accelerations present

↓

$\Gamma_{bc}{}^{a}$ cannot vanish everywhere (in Minkowski space this would be possible)

↓

THE GRAVITATIONAL FIELD WILL BE REPRESENTED BY THE FACT THAT SPACETIME IS CURVED

→ The gravitational field has been "geometrized"

Road to GR

- Consequences:

3. The non-existence of global inertial frames in GR

Recall: By definition in an IF the inertial accelerations should vanish

↓

We then would only have gravitational accelerations present

↓

This contradicts the EP

Remark: In this sense acceleration has lost its absolute meaning in GR
(Compare this to SR, there only velocity has lost its absolute meaning)

↓

The name General Relativity therefore seems to be an appropriate name for the theory

Immediate physical consequences:

1. Light deflected in the gravitational field (which couples to everything)
2. Gravitational redshift

Field equations

- Ansatz: $F_{ab} = \kappa T_{ab}$

To be specified!

coupling constant

Tensorial source (10) (analogy to EM tensor in SR)

$F \underset{\text{EP}}{\sim} R \underset{\text{Geometry}}{\sim} (g, \partial g, \partial^2 g)$

Requirement: FEQ should have Newtonian limit

→ F_{ab} should only have derivatives of g_{ab} up to the 2nd order and should be linear in the 2nd derivatives

Result from Riemannian geometry →

$$F_{ab} = AR_{ab} + Bg_{ab}R + Cg_{ab} \qquad (A, B, C = \text{const})$$

Only tensor which can be constructed from metric fulfilling (2nd order + linearity)

Field equations

$$F_{ab} = AR_{ab} + Bg_{ab}R + Cg_{ab} = \kappa T_{ab}$$

Simplest choice $A = 0$ leads to $T_{ab} \sim g_{ab}$, turns out to be too restrictive

$$R_{ab} + Bg_{ab}R + Cg_{ab} = \kappa T_{ab}$$

2nd simplest choice $B = C = 0$ leads to

$$R_{ab} = \kappa T_{ab}$$

This form was considered by Einstein, in empty space $R_{ab} = 0$ which is correct, but….

Field equations

But from SR we have $T^{ab}{}_{,b} = 0$ which translates in general CS into $T^{ab}{}_{;b} = 0$ but this leads to the requirement that

→ $R^{ab}{}_{;b} = 0$ 4 additional equations

→ 10 + 4 equations for 10 quantities g_{ab}

We already know that $G_{ab} := R_{ab} - \frac{1}{2} g_{ab} R$ fulfills $G^{ab}{}_{;b} = 0$

$$R_{ab} - \frac{1}{2} g_{ab} R + C g_{ab} = G_{ab} + \Lambda g_{ab} = \kappa T_{ab}$$

FEQ GR

so-called cosmological constant

Field equations

Task: Determine coupling constant (FEQ should reduce to Poisson eq. in weak + static situation)

Ansatz:

$$g_{00} \approx \left(1 - \frac{2\phi}{c^2}\right)$$

$T_{00} \sim \rho$ (non-relativistic matter)

$$\nabla^2 \phi = -\frac{1}{2} c^2 \kappa \rho$$

$$\kappa = \frac{8\pi G}{c^2} \qquad [\kappa] = \frac{m}{kg}$$

$$\kappa \approx 1.8657 \times 10^{-26} \frac{m}{kg}$$

Field equations

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}$$

Remarks:

- Covariant by construction
- Set of non-linear partial differential equations
  -> no superposition of solutions
- Exact solutions only for highly symmetric situations
- Different approximation methods exist
  -> in particular for weak-fields
- Cosmological constant historically kept to obtain a closed static solution (with homogeneous distribution of matter)

Field equations

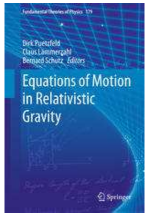
$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}$$

Remarks:

• Note the conceptual difference compared to Maxwell's theory

(in particular consequences of $\nabla_b T^{ab} = 0$)

Further reading and many references →



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Linearized field equations

Assumptions:

1. Exist class of Cartesian-like CS: $|g_{ab} - \eta_{ab}| \ll 1$
2. Asymptotic flatness $g_{ab} \to \eta_{ab} \qquad (r \to \infty)$

$$g_{ab} = \eta_{ab} + \varepsilon h_{ab} + \epsilon^2 h'_{ab} + \cdots$$

$$\longrightarrow \quad g^{ab} = \eta^{ab} - \varepsilon h^{ab}$$

$$h^{ab} := \eta^{ac}\eta^{bd}h_{cd}$$

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Linearized field equations

$$\downarrow$$

$$\Gamma_{ab}{}^{c} = \frac{1}{2}\varepsilon\eta^{cd}\left(h_{db,a} + h_{ad,b} - h_{ab,d}\right)$$

$$\downarrow$$

$$R_{abcd} = \frac{1}{2}\varepsilon\left(h_{bc,ad} + h_{ad,bc} - h_{ac,bd} - h_{bd,ac}\right)$$

$$\downarrow$$

$$R_{ab} = \frac{1}{2}\varepsilon\Box h_{ab} - \frac{1}{2}\varepsilon\eta^{cd}\left(k_{ca,bd} + k_{cb,ad}\right)$$

$$=: \eta^{cd}h_{ab,cd} \qquad k_{ab} := h_{ab} - \frac{1}{2}\eta_{ab}h$$

$$h := \eta^{ab}h_{ab}$$

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Linearized field equations

$$\downarrow$$

$$R = \eta^{ab}R_{ab} = \frac{1}{2}\varepsilon\Box h - \varepsilon\eta^{ac}\eta^{bd}k_{ab,cd}$$

$$\downarrow$$

$$R_{ab} - \frac{1}{2}\eta_{ab}R = \kappa T_{ab}$$

$$\downarrow$$

$$\varepsilon\Box k_{ab} - \varepsilon\eta^{cd}\left(k_{ca,bd} + k_{cb,ad}\right) + \varepsilon\eta_{ab}\eta^{ce}\eta^{df}k_{cd,ef} = 2\kappa T_{ab}$$

$$| \times \partial^b \quad \to \quad \text{LHS} = 0$$

$$\downarrow$$

$$\eta^{bc}T_{ab,c} = 0$$

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Linearized field equations

Consider CT of the form: $\tilde{x}^a = x^a + \varepsilon \xi^a(x^b)$

$\longrightarrow$ $h_{ab} = \tilde{h}_{ab} + 2\varepsilon\xi_{(a,b)}$

$\longrightarrow$ $k_{ab} = \tilde{k}_{ab} + 2\varepsilon\xi_{(a,b)} - \varepsilon\eta_{ab}\xi^c{}_{,c}$

$\xrightarrow{| \times \partial^b}$ $\eta^{bc}k_{ab,c} = \eta^{bc}\tilde{k}_{ab,c} + \epsilon\eta_{ab}\Box\xi^b$

$\downarrow$ Choose ξ^a for given k_{ab} such that $\eta^{bc}k_{ab,c} = \epsilon\eta_{ab}\Box\xi^b$ holds (always possible, sol. of inh. wave eq.)

$$\eta^{bc}\tilde{k}_{ab,c} = 0$$

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Linearized field equations

$\downarrow$

$$\varepsilon\Box\tilde{k}_{ab} = 2\kappa T_{ab}$$

Remarks:

1. Note the analogy with Maxwell's equations in Lorentz gauge
2. Solution can be written down immediately for known source (i.e. retarded solution, over Minkowskian past light cone)

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